

ECE 313: Solutions to Problem Set 5

1. (a) Let's say $\hat{n}_{\text{ML}} < 10$. Then the set of values that $\mathcal{X}$ can take on does not include 10 or any values higher than that, so it is impossible to obtain 10 as the value of $\mathcal{X}$, as it is not in the alphabet $\{1, 2, \dots, \hat{n}_{\text{ML}}\}$.
- (b) The likelihood function is $f(n) = \frac{20}{n(n+1)}$ which is a decreasing function of n . The maximum thus occurs at the minimum value for n . Thus, $\hat{n}_{\text{ML}} = 10$.
- (c) i. It is easy to compute the values of the likelihood function $f(n) = \frac{2(n+1-10)}{n(n+1)}$ for $n \geq 10$. The maximum value is $0.05263158\dots = \frac{1}{19}$ for $n = 18$ and also for $n = 19$. Hence, we conclude $\hat{n}_{\text{ML}}$ is one of $\{18, 19\}$.
- ii. We look at the ratio $\frac{f(n)}{f(n-1)} = \frac{\frac{2(n+1-i)}{n(n+1)}}{\frac{2(n-i)}{(n-1)n}} = \frac{n^2 - ni + i - 1}{n^2 - ni + n - i}$ which is greater than 1, equal to 1, and less than 1, according as $i - 1$ is greater than, equal to, or less than $n - i$, that is, according as n is less than, equal to, or greater than $2i - 1$. In other words,

$$f(i) < f(i+1) < f(i+2) < \dots < f(2i-2) = f(2i-1) > f(2i) > f(2i+1) > \dots$$

so that if we observe that $\mathcal{X} = i$, then the maximum-likelihood estimate of n is $\hat{n}_{\text{ML}} = 2i - 2$ or $\hat{n}_{\text{ML}} = 2i - 1$. For $i = 10$, this result matches the one we obtained by computation.

Notice the difference in the maximum-likelihood estimates of n for the increasing-ramp and decreasing-ramp pmfs. If we observe that $\mathcal{X} = i$, we estimate $\hat{n}_{\text{ML}} = i$ for increasing-ramp pmfs, while for decreasing-ramp pmfs, the maximum-likelihood estimate is $\hat{n}_{\text{ML}} = 2i - 2$ or $\hat{n}_{\text{ML}} = 2i - 1$, nearly twice the observed value of $\mathcal{X}$.

2. (a) We are modeling the guesses as independent trials, and we are not allowing for other possibilities such as on some questions, the student can eliminate one or more alternatives and thus improve chances of getting the right answer to $\frac{1}{4}$ or $\frac{1}{3}$ etc.
- (b) The likelihood that we observe $\mathcal{W} = n$ is $\binom{N-K}{n} (0.8)^n (0.2)^{N-K-n}$ for $n = 0, 1, \dots, N-K$.
- (c) For given N and n , the likelihood of part (b) is a function, say $f(K)$ of K . We have that

$$\frac{f(K)}{f(K-1)} = \frac{\binom{N-K}{n} (0.8)^n (0.2)^{N-K-n}}{\binom{N-(K-1)}{n} (0.8)^n (0.2)^{N-(K-1)-n}} = \frac{N - (K-1) - n}{0.2(N - (K-1))} \geq 1 \text{ iff } K \leq N - 1.25n + 1.$$

Thus, the likelihood is maximum when K has value $\hat{K}_{\text{ML}} = \lfloor N - 1.25n + 1 \rfloor$.

- (d) If n is *not* a multiple of 4, then the analysis of part (b) shows that $f(\hat{K}_{\text{ML}})$ is the unique maximum of the likelihood, and the examiner's estimate $\hat{K}_{\text{GP}} = N - n - \lfloor 0.25n \rfloor$ equals the maximum likelihood estimate. When n is a multiple of 4, then $f(\hat{K}_{\text{ML}}) = f(\hat{K}_{\text{ML}} - 1)$, and thus both $\hat{K}_{\text{ML}}$ and $\hat{K}_{\text{ML}} - 1$ are legitimate maximum likelihood estimates of the unknown quantity. Note that in this case, the examiner's estimate $\hat{K}_{\text{GP}} = \hat{K}_{\text{ML}} - 1$ and thus the examiner is using a maximum likelihood estimate (she is just a tough grader!). For $N = 100$ and $K = 90$, $\mathcal{W}$ can take on values $0, 1, \dots, 10$. The value of $\mathcal{W}$ most likely to occur is 8. In this case, $\hat{K}_{\text{ML}} = 91$ while $\hat{K}_{\text{GP}} = 90$ and so the examiner *does* estimate K correctly. On the other hand, if $\mathcal{W} = 4$, the examiner estimates K to be $\hat{K}_{\text{GP}} = 95$ (erring on the side of caution), and if $\mathcal{W} = 10$, then $\hat{K}_{\text{GP}} = 88$, ouch!
3. (a) This is sampling with replacement and we have that $P(R_1) = P(R_2) = \frac{r}{r+g}$.
- (b) $P(R_2) = P(R_2|R_1)P(R_1) + P(R_2|R_1^c)P(R_1^c) = \frac{r+c}{r+c+g} \times \frac{r}{r+g} + \frac{r}{r+c+g} \times \frac{g}{r+g} = \frac{r}{r+g}$ just as before, *and independent of the value of c!*

$$(c) P\{r + c \text{ red balls} | R_2\} = \frac{P(R_1 \cap R_2)}{P(R_2)} = \frac{P(R_2 | R_1)P(R_1)}{P(R_2)} = P(R_2 | R_1) = \frac{r + c}{r + c + g}$$

4. The events F , C , and S are mutually exclusive and $F \cup C \cup S = \Omega$. Hence, $P(F) + P(C) + P(S) = 1$, and since $P(F) = \frac{1}{2}P(C)$, we get $P(S) = 1 - \frac{3}{2}P(C)$. We know by from the theorem of total probability (Eq. 3.4 in the book) that

$$P(H) = P(C)P(H|C) + P(F)P(H|F) + P(S)P(H|S)$$

where the conditional probabilities are given, and $P(F)$ and $P(S)$ can be expressed in terms of $P(C)$. Solving, we get $P(C) = \frac{5}{12}$.

5. (a) $P(A|B^c) = 1 - P(A^c|B^c) = 0.6$.
 $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = 0.3 \times 0.7 + 0.6 \times 0.3 = 0.39$.
 $P(B|A) = P(A|B)P(B)/P(A) = 0.3 \times 0.7/0.39 = 7/13$.
- (b) $P(E \cap F) = P(F|E)P(E) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$ and $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/8}{1/3} = \frac{1}{3}$
 $\Rightarrow P(F) = \frac{3}{8}$.
- (c) $P(G \cup H) = P(G) + P(H) - P(G \cap H) = \frac{4}{3} - P(G \cap H) \leq 1$ from which we deduce that
 $P(G \cap H) \geq \frac{1}{3}$ and hence $P(G|H) = \frac{P(G \cap H)}{P(H)} \geq \frac{1/3}{2/3} = \frac{1}{2}$.
6. (a) When you first pick, all doors are equally likely, so $P(A) = \frac{1}{3}$.
(b) For the stay-put strategy, $P(B|A) = 1$, and $P(B|A^c) = 0$. Using Equation 3.4 in book, $P(B) = P(A)P(B|A) + P(A^c)P(B|A^c) = \frac{1}{3}$.
(c) For the always-switch strategy, $P(B|A) = 0$, $P(B|A^c) = 1$, and so we get $P(B) = \frac{2}{3}$.
(d) If we pick with equal probabilities $\frac{1}{2}$ one of the two unopened doors, we have that $P(B|A) = P(B|A^c) = \frac{1}{2}$, and so $P(B) = \frac{1}{2}$.
Note that Monty's offer is essentially "Would you rather have what is behind the two doors that you didn't pick (except you don't have to take the goat that has already been revealed home with you), or would you rather stay with your original pick where you have a 1/3 chance of having picked the right door?" Clearly, the always-switch strategy is best.
- (e) Picking doors in turn is essentially sampling without replacement, and so, as we have seen several times (e.g. probability that the i -th key tried by the professor is the right key is $\frac{1}{n}$ for all i), it doesn't matter who picks first: both you and your friend have probability $\frac{1}{3}$ of having chosen the door with the car. Note that the probability is $\frac{2}{3}$ that one of the two doors that you two have picked conceals the car, and $\frac{1}{3}$ that the remaining door hides the car. If your friend has been sent back to his seat, you should stay put to maximize your chances of getting the car. Monty's offer is essentially "Would you rather have what's behind both the two doors that have been picked (except that you don't have to take the goat that has been revealed home with you) or would you like to switch to the remaining door which has 1/3 probability of concealing the car?"
7. This game is different from the one in Problem 6 in that you have no idea what the rules of the game are. If the man is intent on separating you from your money as quickly as possible, he will not offer the chance to switch unless you picked the shell hiding the pea in the first place! That is, if you picked the wrong shell, the man will reveal the pea and you will lose your bet. Of course, if you look like a person willing to play several rounds, the man may set you up by playing by Monty's rules (and allowing you to win with probability 2/3) for some time. Then you will place a large bet on the wrong shell, and all of a sudden, you will not be given the choice of changing your bet! Personally, I would stick with the shell originally chosen since it gives me at least a 1/3 probability of winning regardless of the man's strategy; your experience (and monetary losses) may vary!