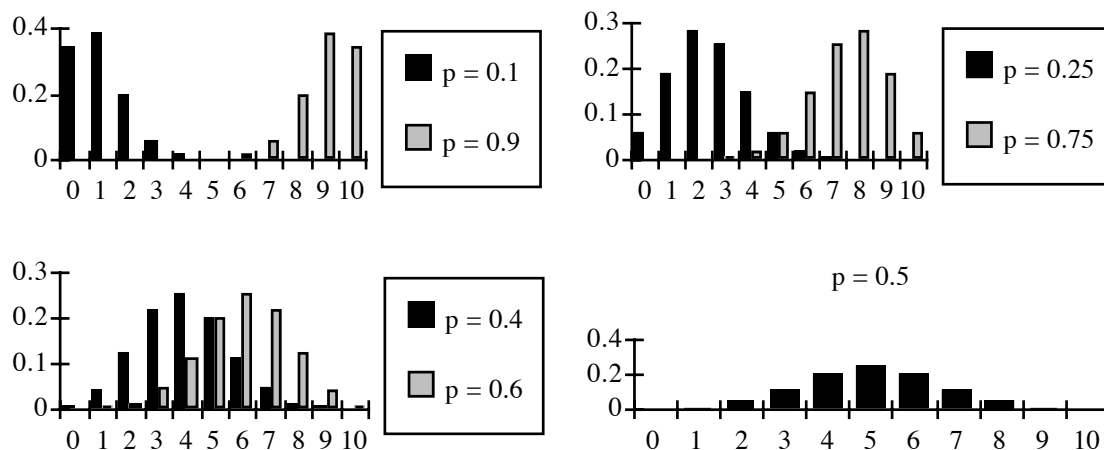


ECE 313: Solutions to Problem Set 4

1. (a) (b) The graphs are as shown below.



- (c) The pmfs of $\mathcal{X}_p$ and $\mathcal{X}_{1-p}$ are “reverses” of each other. For each choice of k , $k = 0, 1, \dots, 10$, and any value of p , $P\{\mathcal{X}_p = k\} = P\{\mathcal{X}_{1-p} = 10 - k\}$. Note that $\mathcal{X}_p$ counts the number of occurrences on 10 independent trials of an event A of probability p . But, if $\mathcal{X}_p = k$, then the *complementary event* A^c (of probability $1 - p$), must have occurred $10 - k$ times, and the number of occurrences of A^c is counted by $\mathcal{X}_{1-p}$.
2. (a) $\mathcal{X}$ denotes a binomial random variable with parameters (N, p) . It counts the number of occurrences of an event A of probability p on N independent trials. $\mathcal{Y} = N - \mathcal{X}$ counts the number of occurrences of A^c , an event of probability $1 - p$, on the N independent trials and is thus a binomial random variable with parameters $(N, 1 - p)$.
- (b) $P\{\mathcal{X} \text{ is even}\} = \binom{N}{0}(1-p)^N + \binom{N}{2}(1-p)^{N-2}p^2 + \binom{N}{4}(1-p)^{N-4}p^4 + \dots$
 Now, a minor extension to Problem 1(e) of Problem Set 2 shows that
 $(x + y)^N + (x - y)^N = 2 \left[\binom{N}{0}x^N + \binom{N}{2}x^{N-2}y^2 + \dots \right]$, and so, setting $x = 1 - p$,
 $y = p$, we get $P\{\mathcal{X} \text{ is even}\} = \frac{1}{2} [(1 - p + p)^N + (1 - p - p)^N] = \frac{1}{2} [1 + (1 - 2p)^N]$.
 Notice that the probability is $1/2$ for $p = 1/2$.
3. $\mathcal{X}$ takes on values $0, 1, 2, 3, 4, 5, 6, 7, 8$ with probabilities $p_{\mathcal{X}}(i) = \frac{1}{256}, \frac{8}{256}, \frac{28}{256}, \frac{56}{256}, \frac{70}{256}, \frac{56}{256}, \frac{28}{256}, \frac{8}{256}, \frac{1}{256}$ respectively.

(a) $E[\mathcal{X}] = \sum_{i=0}^8 i \cdot p_{\mathcal{X}}(i) = 4$ by tedious calculation, or, more simply, $E[\mathcal{X}] = np = 4$.

(b) $\mathcal{Y}$ takes on values $1, 2, 3, 0$ with probabilities $\frac{28}{256}, \frac{8}{256}, \frac{1}{256}$, and $1 - \frac{28+8+1}{256} = \frac{219}{256}$.

(c) i. $E[\mathcal{Y}] = \sum_{i=0}^3 i \cdot p_{\mathcal{Y}}(i) = \frac{47}{256}$. ii. $E[\mathcal{Y}] = \sum_{i=0}^8 \max\{\mathcal{X} - 5, 0\} \cdot p_{\mathcal{X}}(i) = \frac{47}{256}$

- (d) The airline has collected $8F$ in fares from the passengers. The operating cost is $200 + 10(\mathcal{X} - \mathcal{Y})$ and the airline pays out $\$(F + 20)\mathcal{Y}$ in returned fares plus penalties. Thus, $\mathcal{Z} = 8F - 200 - 10(\mathcal{X} - \mathcal{Y}) - (F + 20)\mathcal{Y} = 8F - 200 - 10\mathcal{X} - (F + 10)\mathcal{Y}$ is the net profit from the flight. Hence, $E[\mathcal{Z}] = 8F - 200 - 10E[\mathcal{X}] - (F + 10)E[\mathcal{Y}] = 8F - 240 - (F + 10)\frac{47}{256}$ is the average profit from this flight, which is 0 for $F \approx \$30.9395\dots$. Note that on average, the airline keeps 7.81... of the 8 fares that it collects and the average operating cost of the flight is only \$238.16.... Including the penalties for denied boarding only increases the average cost to \$241.83....
- (e) $\mathcal{Z}$ takes on values $8F - 200, 8F - 210, 8F - 220, 8F - 230, 8F - 240, 8F - 250, 7F - 270, 6F - 290$, and $5F - 310$ with probabilities $\frac{1}{256}, \frac{8}{256}, \frac{28}{256}, \frac{56}{256}, \frac{70}{256}, \frac{56}{256}, \frac{28}{256}, \frac{8}{256}, \frac{1}{256}$ respectively. For $F = 30.9395\dots$, the values taken on by $\mathcal{Z}$ are 47.516..., 37.516..., 27.516..., 17.516..., 7.516..., -2.48..., -53.42..., -104.36... and -155.30.... Thus, the airline loses money when more than 4 passengers show up but these losses occur with small probability, and the average profit is 0 (Verify this!)
4. (a) When the professor is putting unsuccessful keys back in his pocket and thus sampling with replacement, then on any given trial the probability of success is $p = \frac{1}{n}$ and the probability of failure is $q = (1 - p) = 1 - \frac{1}{n}$. Thus, $P\{\mathcal{X} = i\} = p \cdot q^{i-1}$ for $i = 1, 2, 3, \dots$ and $\mathcal{X}$ is a *geometric* random variable with parameter $p = \frac{1}{n}$. Hence, $E[\mathcal{X}] = p^{-1} = n$. Alternatively, we can use the series developed in Problem 2(d) of Problem Set 1 to write

$$E[\mathcal{X}] = \sum_{i=1}^n i \cdot p_{\mathcal{X}}(i) = p + 2pq + 3pq^2 + \dots = p(1 + 2q + 3q^2 + \dots) = \frac{p}{(1 - q)^2} = \frac{p}{p^2} = \frac{1}{p} = n.$$

- (b) When the professor is putting unsuccessful keys in his briefcase and thus sampling (from his pocket) without replacement, $\mathcal{X}$ takes on values $1, 2, \dots, n$ only with equal probability $\frac{1}{n}$. To see this, note that the keys can be thought of as being arranged in his pocket in one of $n!$ equally likely orderings, and he takes the keys out of his pocket in the order in which they are arranged. Exactly $(n - 1)!$ of the $n!$ possible orderings have the right key as the i -th one to be tried (because the $n - 1$ wrong keys occur in the $n - 1$ other spots in $(n - 1)!$ ways). Hence $P\{\mathcal{X} = i\} = \frac{(n-1)!}{n!} = \frac{1}{n}$ for $i = 1, 2, \dots, n$ and

$$E[\mathcal{X}] = \sum_{i=1}^n i \cdot p_{\mathcal{X}}(i) = \sum_{i=1}^n i \cdot \frac{1}{n} = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}.$$

Notice that the obvious stratagem of putting unsuccessful keys aside instead of putting them back in the pocket nearly halves the expected number of tries.

5. (a) $E[\mathcal{X}] = \sum_{k=0}^{\infty} k \cdot p_{\mathcal{X}}(k) = 0 \cdot p_{\mathcal{X}}(0) + 1 \cdot p_{\mathcal{X}}(1) + 2 \cdot p_{\mathcal{X}}(2) + 3 \cdot p_{\mathcal{X}}(3) + 4 \cdot p_{\mathcal{X}}(4) + \dots$
 $= p_{\mathcal{X}}(1)$
 $+ p_{\mathcal{X}}(2) + p_{\mathcal{X}}(2)$
 $+ p_{\mathcal{X}}(3) + p_{\mathcal{X}}(3) + p_{\mathcal{X}}(3)$
 $+ p_{\mathcal{X}}(4) + p_{\mathcal{X}}(4) + p_{\mathcal{X}}(4) + p_{\mathcal{X}}(4)$
 $+ \dots$

The first displayed column sums to $P\{\mathcal{X} > 0\}$, the second to $P\{\mathcal{X} > 1\}$, the third to $P\{\mathcal{X} > 2\}$ and so on. Hence we get that $E[\mathcal{X}] = \sum_{i=0}^{\infty} P\{\mathcal{X} > i\}$.

- (b) For a geometric random variable with parameter p ,

$$\begin{aligned} P\{\mathcal{X} > i\} &= (1-p)^i p + (1-p)^{i+1} p + (1-p)^{i+2} p + \cdots = (1-p)^i p \sum_{k=0}^{\infty} (1-p)^k \\ &= (1-p)^i p \times \frac{1}{1-(1-p)} = (1-p)^i. \end{aligned}$$

More directly, the event $\{\mathcal{X} > i\}$ occurs if and only if the first i trials ended in failure, and this has probability $(1-p)^i$.

$$E[\mathcal{X}] = \sum_{i=0}^{\infty} P\{\mathcal{X} > i\} = \sum_{i=0}^{\infty} (1-p)^i = \frac{1}{1-(1-p)} = \frac{1}{p} \text{ as before.}$$

6. (a) $P\{\mathcal{Y} \text{ is even}\} = P\{\mathcal{Y} = 0\} + P\{\mathcal{Y} = 2\} + P\{\mathcal{Y} = 4\} + \cdots = \exp(-\lambda) \left[1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots \right]$
 $= \exp(-\lambda) \cosh(\lambda)$ by recognizing the series in square brackets.
- (b) $\frac{1}{2} [1 + (1-2p)^N] = \frac{1}{2} \left[1 + \left(1 - \frac{2\lambda}{N} \right)^N \right] \rightarrow \frac{1}{2} [1 + \exp(-2\lambda)] = \exp(-\lambda) \frac{\exp(\lambda) + \exp(-\lambda)}{2}$
 $= \exp(-\lambda) \cosh(\lambda).$
- (c) The likelihood that we observed $\mathcal{Y} = k$ is $\exp(-\lambda) \lambda^k / k!$. This has derivative $[-\exp(-\lambda) \lambda^k + k \exp(-\lambda) \lambda^{k-1}] / k! = 0$ at $\lambda = k$. Thus, the maximum-likelihood estimate of λ when $\mathcal{Y} = k$ is observed is $\hat{\lambda} = k$. Note that for a binomial random variable, the maximum likelihood estimate of p is k/n , i.e., the maximum-likelihood estimate of np is k , and recall that the Poisson random variable with parameter $\lambda = np$ approximates the binomial random variable. Therefore, our above result is consistent with the previous one.
7. (a) On average, $E[\mathcal{X}] = 105 \times 0.9 = 94.5$ passengers show up for the flight.
- (b) $P\{\mathcal{X} \leq 100\} = 1 - P\{\mathcal{X} > 100\}$
 $= 1 - P\{\mathcal{X} = 101\} - P\{\mathcal{X} = 102\} - P\{\mathcal{X} = 103\} - P\{\mathcal{X} = 104\} - P\{\mathcal{X} = 105\}$
 $= 1 - \binom{105}{101} (0.9)^{101} (0.1)^4 - \binom{105}{102} (0.9)^{102} (0.1)^3 - \binom{105}{103} (0.9)^{103} (0.1)^2 - \binom{105}{104} (0.9)^{104} (0.1)^1 -$
 $\binom{105}{105} (0.9)^{105} (0.1)$
 $= 1 - \binom{105}{4} (0.9)^{101} (0.1)^4 - \binom{105}{3} (0.9)^{102} (0.1)^3 - \binom{105}{2} (0.9)^{103} (0.1)^2 - \binom{105}{1} (0.9)^{104} (0.1)^1 -$
 $\binom{105}{0} (0.9)^{105} (0.1)$
 $= 0.9832 \dots$
- (c) We saw earlier that if $\mathcal{X}$ is a binomial random variable with parameters (n, p) , then $\mathcal{Y} = n - \mathcal{X}$ is a binomial random variable with parameters $(n, 1-p)$.
- (d) $P\{\mathcal{Y} \geq 5\} = 1 - P\{\mathcal{Y} = 0\} - P\{\mathcal{Y} = 1\} - P\{\mathcal{Y} = 2\} - P\{\mathcal{Y} = 3\} - P\{\mathcal{Y} = 4\}$
 $= 1 - \exp(-10.5) \left[1 + \frac{10.5}{1!} + \frac{(10.5)^2}{2!} + \frac{(10.5)^3}{3!} + \frac{(10.5)^4}{4!} \right] = 0.9789 \dots$ which is close enough to 0.9832... for gummint work.