

## ECE 313: Solutions to Problem Set 2

$$1. \quad (1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \binom{n}{4}x^4 + \dots$$

$$(1-x)^n = \binom{n}{0} - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 + \binom{n}{4}x^4 - \dots$$

from which we get that  $(1+x)^n + (1-x)^n = 2 \left[ \binom{n}{0} + \binom{n}{2}x^2 + \binom{n}{4}x^4 + \dots \right]$ . Set  $x = 1$  and note that the left side has value  $2^n$  while the right side is twice the number of sets with an even number of elements. More simply, set  $x = 1$  in  $(1-x)^n$  to get

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots \Rightarrow \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \dots = \binom{n}{1} + \binom{n}{3} + \dots = 2^{n-1}.$$

2. (a) For any event  $A$ ,  $P(A) \leq 1$ . Hence,  $P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1$ . It follows that

$$P(A \cap B) \geq P(A) + P(B) - 1$$

- (b) i. Since  $A$  is a subset of  $A \cup B$ , we know that  $P(A) \leq P(A \cup B)$ . Similarly  $P(B) \leq P(A \cup B)$ . Adding the two inequalities gives that  $P(A) + P(B) \leq 2P(A \cup B)$ , and thus it follows that

$$\frac{P(A) + P(B)}{2} \leq P(A \cup B)$$

with equality iff  $P(A) = P(B) = P(A \cup B)$  or equivalently,  $P(A) = P(B) = P(A \cap B)$

To prove the other inequality, note that  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$  for any events  $A$  and  $B$ . Since  $P(A \cap B) \geq 0$ , it follows that

$$P(A \cup B) \leq P(A) + P(B)$$

with equality iff  $P(A \cap B) = 0$ . This bound is sometimes referred to as the *union bound*.

- ii. As shown in the previous part,  $P(A) \leq P(A \cup B \cup C)$ ,  $P(B) \leq P(A \cup B \cup C)$  and  $P(C) \leq P(A \cup B \cup C)$ . Adding the three inequalities gives that  $P(A) + P(B) + P(C) \leq 3P(A \cup B \cup C)$ , and thus it follows that

$$\frac{P(A) + P(B) + P(C)}{3} \leq P(A \cup B \cup C)$$

with equality iff  $P(A) = P(B) = P(C) = P(A \cup B \cup C)$ .

In the previous part, we proved that  $P(A \cup B) \leq P(A) + P(B)$  holds for events  $A$  and  $B$ . Let  $D$  denote  $A \cup B$ . Then,

$$P(A \cup B \cup C) = P(D \cup C) \leq P(D) + P(C) = P(A \cup B) + P(C) \leq P(A) + P(B) + P(C).$$

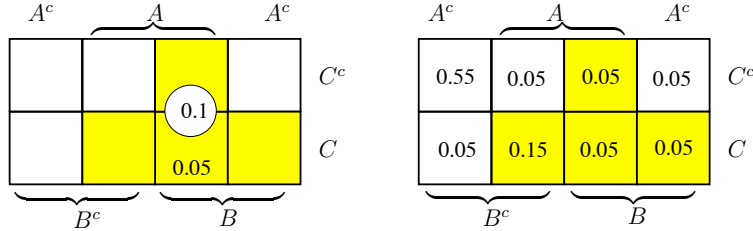
Equality holds iff  $P(A \cap B) = P(A \cap C) = P(B \cap C) = 0$  (which implies that  $P(A \cap B \cap C) = 0$ .)

3. We have two fair dice, and they are rolled independently. Since they are fair, the probability of either die landing with any particular side up is  $\frac{1}{6}$ . Denote the events as  $R_i, B_i, Y_i$ , and  $W_i$  be the events that die  $i$  landed with a face up of color red, black, yellow, and white respectively. So the probability we are looking for is  $Pr(R_1 R_2) + Pr(B_1 B_2) + Pr(Y_1 Y_2) + Pr(W_1 W_2)$ . Since the die are rolled independently,  $Pr(R_1 R_2) = Pr(R_1)Pr(R_2)$  and similarly for the other colors.  $Pr(R_i) = Pr(B_i) = \frac{1}{3}$  and  $Pr(Y_i) = Pr(W_i) = \frac{1}{6}$ . So the answer is  $\frac{5}{18} = 0.2778$ .

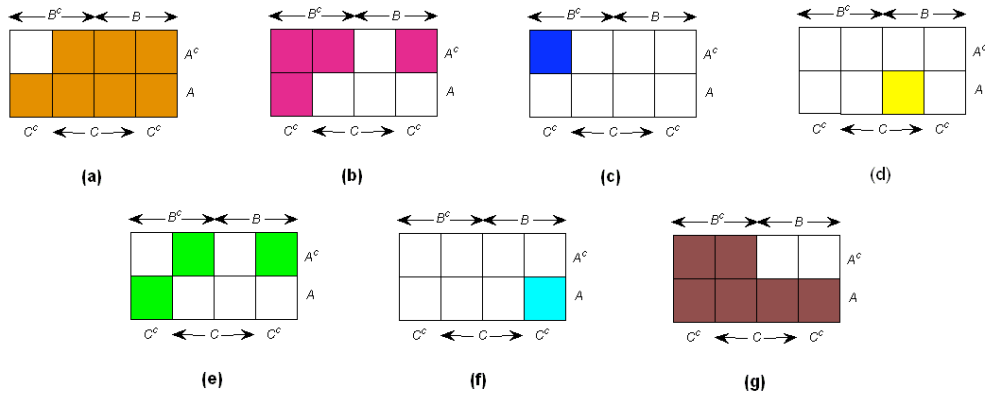
4. To help us with this, first let us simplify the given expression. So we have  $P(A \cup (B^c \cup C^c)^c)$  where  $B^c$  denotes the complement of event  $B$ .  $(B^c \cup C^c)^c = ((B^c)^c \cap (C^c)^c) = (B \cap C)$ . So the original expression reduces to  $P(A \cup (B \cap C)) = P((A \cup B) \cap (A \cup C))$ .

- (a) Since the events are mutually exclusive,  $B \cap C = \emptyset$  so  $P(A \cup (B \cap C)) = P(A \cup \emptyset) = P(A) = \frac{1}{3}$ .

- (b) Remembering that  $P(D \cup E) = P(D) + P(E) - P(D \cap E)$ , and letting  $D = A$  and  $E = B \cap C$ , we get  $P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C)) = \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$
- (c) We can use the previous expression, noting that since  $P(A \cap C) = 0$ ,  $P(A \cap (B \cap C)) = 0$ , so  $P(A \cup (B \cap C)) = P(A) + P(B \cap C) - P(A \cap (B \cap C)) = \frac{1}{2} + \frac{1}{3} - 0 = \frac{5}{6}$
- (d) Working with what we have, take complement.  $P(A \cup (B \cap C)) = 1 - P([A \cup (B \cap C)]^c) = 1 - P(A^c \cap (B \cap C)^c) = 1 - P(A^c \cap (B^c \cup C^c)) = 1 - 0.6 = 0.4$
5. (a) The Karnaugh map is as shown in the left hand figure below, with some probabilities marked on it. Note that the shaded region is the event  $(A \cap B) \cup (B \cap C) \cup (A \cap C)$ .



- (b) Since  $A \cap B$  is the disjoint union of  $A \cap B \cap C$  and  $A \cap B \cap C^c$ , we get that  $P(A \cap B) = 0.1 = P(A \cap B \cap C) + P(A \cap B \cap C^c) = 0.05 + P(A \cap B \cap C^c)$  giving that  $P(A \cap B \cap C^c) = 0.05$ . Since  $P(AB \cup BC \cup AC) = 0.3 = P(AB) + P(AB^cC) + P(A^cBC)$  while  $P(AC) = P(ABC) + P(AB^cC) = 2P(BC) = P(ABC) + P(A^cBC)$ , we readily obtain that  $P(AB^cC) = 0.15$  and  $P(A^cBC) = 0.05$ . Since  $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6$  we have that  $P(A^cB^c) = 1 - P(A \cup B) = 0.4$ . Since  $P(A^cB^cC) = P(C) - P(BC) - P(AB^cC) = 0.05$ , we get that  $P(\text{cereal snaps, crackles, and pops}) = P(A^c \cap B^c \cap C) = 0.55$ .
- (c)  $P(\text{the sample fails only the snap test}) = P(AB^cC^c) = 0.05$ .  
 $P(\text{the sample fails only the crackle test}) = P(A^cBC^c) = 0.05$ .  
 $P(\text{the sample fails only the pop test}) = P(A^cB^cC) = 0.05$ .
6. (a) At least one of the events  $A, B, C$  occurs;  $- A \cup B \cup C = (A^cB^cC^c)^c$
- (b) At most one of the events  $A, B, C$  occurs;  $- (AB^cC^c) \cup (A^cBC^c) \cup (A^cB^cC) \cup (A^cB^cC^c) = (AB \cup BC \cup AC)^c$
- (c) None of the events  $A, B, C$  occurs;  $- (A \cup B \cup C)^c = A^cB^cC^c$  [compare to part (a)]
- (d) All three events  $A, B, C$  occur;  $- ABC$
- (e) Exactly one of the events  $A, B, C$  occurs;  $- (AB^cC^c) \cup (A^cBC^c) \cup (A^cB^cC)$
- (f) Events  $A$  and  $B$  occur, but not  $C$ ;  $- ABC^c$
- (g) Either event  $A$  occurs, or if not then  $B$  also does not occur;  $- A \cup A^cB^c = A \cup B^c$



7. (a) There are  $\binom{5}{2} = 10$  games in this tournament, and  $\binom{n}{2} = \frac{n(n-1)}{2}$  games in general.
- (b) Yes. In fact, many schedules can be specified. Here is one: imagine that the teams have been arranged in a circle so that each team can be thought of as having two teams on its left and two teams on its right. Then, each team wears home uniforms when playing a team on its left and away uniforms when playing a team on its right.
- (c) There are  $2^{10} = 1024$  possible outcomes of this tournament.
- Two teams cannot possibly have 4-0 records. However, if Team A, say, has a 4-0 record, then we know what happened in 4 games, while the outcomes of the remaining 6 games are arbitrary. Hence,  $P(\text{Team A has a 4-0 record}) = \frac{2^6}{2^{10}} = \frac{1}{2^4} = \frac{1}{16}$  and  
 $P(\text{some team has a 4-0 record}) = 5 \times \frac{1}{16} = \frac{5}{16}$ . (What axiom are we using here?)
  - The same argument shows that  $P(\text{some team has a 0-4 record}) = \frac{5}{16}$ .
  - If Team A wins all four of its games *and* Team B loses all four of its games, then we know what happened in 7 games (why not 8?), and hence we get that  
 $P(\text{Team A has 4-0 record; Team B has a 0-4 record}) = \frac{2^3}{2^{10}} = \frac{1}{2^7} = \frac{1}{128}$ , and  
 $P(\text{some team has 4-0 record; some other team has a 0-4 record}) = 20 \times \frac{1}{128} = \frac{5}{32}$ .
  - The remaining teams have lost one game (against the 4-0 team) and won another game (against the 0-4 team), and will have identical 2-2 records if each wins one game and loses one game among the three games that these teams play against one another (e.g. A beats B who beats C who beats A: basketball is not necessarily a transitive game!). Since only 2 of the 8 outcomes of these three games give 2-2 records for all three teams, we get that  
 $P(\text{one team is 4-0; another is 0-4; rest are 2-2}) = \frac{5}{32} \times \frac{2}{8} = \frac{5}{128}$ .