

ECE 313: Solutions to Problem Set 1

1. (a) Let n denote a positive integer. Then,

$$(1-x)[1+x+x^2+\cdots+x^{n-1}] = [1+x+x^2+\cdots+x^{n-1}] - [x+x^2+\cdots+x^n] = 1-x^n.$$

If $x \neq 1$, divide both sides by $(1-x)$ to get $1+x+x^2+\cdots+x^{n-1} = \frac{1-x^n}{1-x}$.

- (b) At $x = 1$, the sum is $1+1^1+1^2+\cdots+1^{n-1} = n$. Now, since $\frac{1-x^n}{1-x}$ is the indeterminate form $\frac{0}{0}$ at $x = 1$, we apply L'Hôpital's rule to get $\lim_{x \rightarrow 1} \frac{1-x^n}{1-x} = \lim_{x \rightarrow 1} \frac{-nx^{n-1}}{-1} = n$. Therefore, the value of $1+x+x^2+\cdots+x^{n-1}$ at $x = 1$ does equal $\lim_{x \rightarrow 1} \frac{1-x^n}{1-x}$.

- (c) Since $|x| < 1$, $1+x+x^2+\cdots = \lim_{n \rightarrow \infty} 1+x+x^2+\cdots+x^{n-1} = \lim_{n \rightarrow \infty} \frac{1-x^n}{1-x} = \frac{1}{1-x}$.

- (d) Using the product rule, $\frac{d}{dx} \exp(-x) \sum_{k=0}^n \frac{x^k}{k!} = -\exp(-x) \sum_{k=0}^n \frac{x^k}{k!} + \exp(-x) \sum_{k=1}^n \frac{kx^{k-1}}{k!}$
 $= -\exp(-x) \left(\sum_{k=0}^n \frac{x^k}{k!} - \sum_{j=0}^{n-1} \frac{x^j}{j!} \right) = -\exp(-x) \frac{x^n}{n!}.$

Next, since $\sum_{k=0}^{\infty} \frac{x^k}{k!} = \exp(x)$, we have that $\exp(-x) \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1$ and so its derivative is 0.

- (e) $x^n \exp(-ax)$ is positive on $(0, \infty)$ and approaches 0 at both ends. Hence it has a maximum on $(0, \infty)$ which can be found by equating the derivative to 0. We get that $x^n \exp(-ax)$ has maximum value $\left(\frac{n}{a}\right)^n \exp(-n)$ at $x = n/a$. Here, $n = 25$ and $a = \ln(1.00001)$ giving a maximum value of $0.123365 \dots \times 10^{150}$ at $x = 2500012.5 \dots$.
2. (a) The 0-th derivative $f^{(0)}(x)$ is just $f(x) = (1+x)^n$ itself. The k -th derivative is $f^{(k)}(x) = n(n-1)\cdots(n-k+1)(1+x)^{n-k}$ for $k \leq n$. Note that $f^{(n)}(x) = n(n-1)\cdots 2 \cdot 1 = n!$ is a constant, and hence the higher-order derivatives are 0. Thus, the MacLaurin series for $f(x)$ is

$$f(x) = (1+x)^n = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^n \frac{n(n-1)\cdots(n-k+1)}{k!} x^k.$$

- (b) $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots 2 \cdot 1}{((n-k)(n-k-1)\cdots 2 \cdot 1)k!} = \frac{n(n-1)\cdots(n-k+1)}{k!}$
 and so the results match.

- (c) Using the chain rule, $g^{(k)}(x) = (-n)(-n-1)(-n-2)\cdots(-n-k+1)(1-x)^{-n-k}(-1)^k$
 $= n(n+1)(n+2)\cdots(n+k-1)(1-x)^{-n-k}$. Thus, $g^{(k)}(0) = n(n+1)(n+2)\cdots(n+k-1) \neq 0$ for $k \geq 0$. Hence, the MacLaurin series for $g(x)$ contains terms of all degrees, and we have that

$$g(x) = (1-x)^{-n} = \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{n(n+1)\cdots(n+k-1)}{k!} x^k$$

in which the coefficient of x^{n+1} is $\frac{n(n+1)\cdots(2n)}{(n+1)!}$,

- (d) For $n = 1$, $n(n+1)\cdots(n+k-1) = k!$ while for $n = 2$, $n(n+1)\cdots(n+k-1) = (k+1)!$. Hence,

$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k = 1+x+x^2+\cdots \quad \text{and} \quad (1-x)^{-2} = \sum_{k=0}^{\infty} (k+1)x^k = 1+2x+3x^2+\cdots.$$

- (e) The k -th derivative of $h(x) = (1+x)^a$ where a is a real number and not necessarily an integer is $h^{(k)}(x) = a(a-1)\cdots(a-k+1)(1+x)^{a-k}$ where the exponent cannot equal 0 when a is not an integer. Thus, the MacLaurin series has terms of all degrees and we have that

$$h(x) = (1+x)^a = \sum_{k=0}^{\infty} \frac{h^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{a(a-1)\cdots(a-k+1)}{k!} x^k.$$

3. (a) $\int_{-1}^2 |x| dx = \int_{-1}^0 -x dx + \int_0^2 x dx = \left. \frac{-x^2}{2} \right|_{-1}^0 + \left. \frac{x^2}{2} \right|_0^2 = \frac{1}{2} + 2 = 2.5$. The substitution $y = 1-x$ gives

$$\int_{-2}^1 x(1-x)^{19} dx = \int_3^0 (1-y)y^{19}(-dy) = \int_0^3 y^{19} - y^{20} dy = \frac{3^{20}}{20} - \frac{3^{21}}{21} = -\frac{13 \times 3^{20}}{140}.$$

- (b) No nonnegative function $f(x)$ can satisfy $\int_{-2}^1 f(x) dx < 0$. The comparison test says that if $f(x) \geq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$. Now let $g(x) = 0$. More simply, if the curve $f(x)$ lies above the x axis in an interval $[a, b]$, the area under the curve between a and b cannot be negative.

- (c) Via the chain rule we have: (i) False: $\frac{d}{dx} f(-x) = -g(-x)$. (ii) False: $\frac{d}{dx} f(x^2) = g(x^2) \cdot 2x$. (iii) True. (iv) False: $\frac{d}{dx} \exp(f(x^2)) = \exp(f(x^2))g(x^2) \cdot 2x$. (v) False: the antiderivative of $g(-x)$ is $-f(-x)$. (vi) False: the antiderivative of $g(x^2)$ need not be related to $f(x^2)$ at all.

4. (a) i. $I = \int_{-1}^1 \frac{2}{1+x^2} dx = 2 \cdot \arctan(x) \Big|_{-1}^1 = 2 \cdot \left(\frac{\pi}{4} - \frac{-\pi}{4} \right) = \pi$.

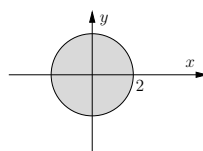
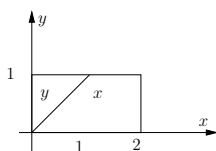
- ii. The substitution $y = 1/x$ changes the integrand to $\frac{2}{1+1/y^2}(-1/y^2)dy = \frac{-2}{1+y^2} dy$ which is the integrand for J . However, as x varies from -1 to 0 , y varies from -1 to $-\infty$. Similarly, as x varies from 0 to 1 , y varies from ∞ to 1 . Therefore, we get

$$\int_{-1}^1 \frac{2}{1+x^2} dx = \int_{-1}^{-\infty} \frac{-2}{1+y^2} dy + \int_{\infty}^1 \frac{-2}{1+y^2} dy = \int_{-\infty}^{-1} \frac{2}{1+y^2} dy + \int_1^{\infty} \frac{2}{1+y^2} dy = \pi.$$

- Thus, the substitution $y = 1/x$ does not change I into J as asserted in the problem statement. iii. No need to re-write the math texts: $\pi \neq 0$.

5. (a) $f(x, y) = \max(x, y)$ takes on values x and y in the regions indicated in the figure below. Hence,

$$\int_{y=0}^1 \int_{x=0}^2 f(x, y) dx dy = \int_{y=0}^1 \int_{x=0}^y y dx dy + \int_{y=0}^1 \int_{x=y}^2 x dx dy = \int_{y=0}^1 y^2 + 2 - y^2/2 dy = \frac{13}{6}.$$



- (b) The integral is over the exterior of a circle of radius 2. Thus, a change to polar coordinates gives

$$\iint_{x^2+y^2>4} (x^2+y^2)^{-2} dx dy = \int_{r=2}^{\infty} \int_{\theta=0}^{2\pi} r^{-4} r d\theta dr = 2\pi \left. \frac{-r^{-2}}{2} \right|_{r=2}^{\infty} = \frac{\pi}{4}.$$

6. (a) $\frac{d}{dx} \exp\left(-\frac{x^2}{2}\right) = -x \exp\left(-\frac{x^2}{2}\right)$.

(b) From part (a), $\int_0^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx = -\exp\left(-\frac{x^2}{2}\right) \Big|_0^{\infty} = 1$.

- (c) The integrand is an odd function and hence the integral has value 0.