University of Illinois Fall 2008

ECE 313: Final Exam

Friday December 12, 2008

1. **[35 points]** A particular webserver may be working (event W) or not working (event W^c). It is known that $P(W) = 0.8$. An attempt to access the webserver can be a failure (event F) or a success (event F^c). Obviously, $P(F|W^c) = 1$, but even if the webserver is working, an attempt to access it can fail due to network congestion. It is known that $P(F^c|W) = 0.9$.

   (a) **[5 points]** Find $P\{\text{access attempt fails}\}$.

   (b) **[10 points]** Find $P\{\text{server is working} \mid \text{access attempt fails}\}$.

   Now, suppose that the results of two consecutive attempts to access the webserver are *conditionally independent* events given that the webserver is working, and also *conditionally independent* given that the webserver is not working.

   (c) **[10 points]** Find $P\{\text{second access attempt fails} \mid \text{first access attempt fails}\}$.

   (d) **[10 points]** Find $P\{\text{server is working} \mid \text{first and second access attempts fail}\}$.

2. **[20 points]** $\mathcal{X}$ denotes a Poisson random variable with parameter $\ln(3)$. Find the numerical values of the mean and variance of $\mathcal{Y} = \cos(\pi\mathcal{X})$.

3. **[15 points]** For what value of C, if any, is $f(u) = \exp(Cu^2), -\infty < u < \infty$, a valid pdf?

4. **[30 points]** $\mathcal{X}$ denotes a *uniform* random variable with mean 1 and variance 3.

   (a) **[10 points]** Find $P\{\mathcal{X} < 0\}$.

   (b) **[10 points]** Find $\mathsf{E}[|\mathcal{X}|]$.

   (c) **[10 points]** Find the pdf of $\mathcal{Y} = |\mathcal{X}|$. In order to receive full credit, you must specify the value of $f_{\mathcal{Y}}(v)$ for all $v, -\infty < v < \infty$.

5. **[40 points]** A radio-frequency signal is either a radar echo (hypothesis H_1) or ambient noise (hypothesis H_0). The *phase* of the signal is modeled as a continuous random variable $\mathcal{X}$ whose pdf is as follows:

   - When H_0 is true, $\mathcal{X}$ has pdf $f_0(u) = \begin{cases} \frac{1}{2\pi}, & -\pi < u < \pi, \\ 0, & \text{elsewhere.} \end{cases}$
   - When H_1 is true, $\mathcal{X}$ has pdf $f_1(u) = \begin{cases} \frac{1}{2\pi}\left(1 + \cos u\right), & -\pi < u < \pi, \\ 0, & \text{elsewhere.} \end{cases}$

   The radar receiver measures $\mathcal{X}$ and decides which hypothesis is true.

   (a) **[10 points]** Suppose that the *maximum-likelihood* decision rule is being used. What value(s) of $\mathcal{X}$ result in a decision in favor of H_1?

   (b) **[10 points]** Find the *false alarm* probability P_{FA} and the *missed detection* or *false dismissal* probability P_{MD} of the maximum-likelihood decision rule.

(c) [**10 points**] Now suppose that $P(\mathsf{H}_0) = \pi_0 = \frac{1}{3}$, $P(\mathsf{H}_1) = \pi_1 = \frac{2}{3}$. What is the *average* error probability $\bar{P}_e$ of the maximum *a posteriori* probability (MAP) (that is, minimum-error-probability or Bayesian) decision rule?

(d) [**10 points**] For what values, if any, of $\pi_0, 0 < \pi_0 < 1$ does the MAP rule *always* decide in favor of H_0 regardless of the value of $\mathcal{X}$?

6. [**35 points**] The joint pdf of random variables $\mathcal{X}$ and $\mathcal{Y}$ is given by

$$f_{\mathcal{X},\mathcal{Y}}(u,v) = \begin{cases} u+v, & 0 < u < 1,\ 0 < v < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) [**15 points**] Find $P\{\mathcal{X} + \mathcal{Y} > 1\}$.

(b) [**20 points**] Find $P\{\mathcal{X}^2 + \mathcal{Y}^2 \leq 1\}$.

7. [**30 points**] $\mathcal{X}$ and $\mathcal{Y}$ are independent random variables with pdfs as specified below:

$$f_{\mathcal{X}}(u) = \begin{cases} \exp(-u), & u > 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_{\mathcal{Y}}(v) = \begin{cases} \exp(v), & v < 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find $f_{\mathcal{Z}}(\alpha)$, the pdf of the random variable $\mathcal{Z} = \mathcal{X} + \mathcal{Y}$. Be sure to specify the value of $f_{\mathcal{Z}}(\alpha)$ for all real numbers α.

8. [**20 points**] $\mathcal{X}$ and $\mathcal{Y}$ are random variables such that $\mathsf{var}(\mathcal{X}) = \mathsf{var}(\mathcal{Y}) = 1$. Suppose that $\mathcal{Q} = 2\mathcal{X} + 3\mathcal{Y} + 4$ and $\mathcal{R} = 3\mathcal{X} - 4\mathcal{Y} + 2$ and that $\mathsf{var}(\mathcal{Q}) = \mathsf{var}(\mathcal{R})$.

(a) [**10 points**] What is the value of the *correlation coefficient* $\rho_{\mathcal{X},\mathcal{Y}}$ and the value of $\mathsf{var}(\mathcal{Q})$?

(b) [**10 points**] Now suppose also that $\mathcal{X}$ and $\mathcal{Y}$ are *jointly Gaussian* random variables. Are $\mathcal{X} + \mathcal{Y}$ and $\mathcal{X} - \mathcal{Y}$ *independent* random variables?