

ECE 313: Solutions to Hour Exam II

1. Consider an experiment in which a fair tetrahedral (that is, four-sided) die is rolled. The faces of the die are marked 1, 2, 3, and 4, and the outcome is defined as the number on the bottom face when the die stops rolling. Thus, the sample space is $\Omega = \{1, 2, 3, 4\}$. Let $E_2 = \{1, 2\}$, $E_3 = \{1, 3\}$, and $E_4 = \{1, 4\}$.

- (a) Are the events E_2 and E_3 (mutually) independent events?
 (b) Are the events E_2 , E_3 , and E_4 pairwise independent?
 (c) Are the three events E_2 , E_3 , and E_4 mutually independent?

Solution: $P(E_2) = P(E_3) = P(E_4) = \frac{1}{2}$.

$E_2E_3 = E_2E_4 = E_3E_4 = E_2E_3E_4 = \{1\}$.

Hence, $P(E_2E_3) = P(E_2E_4) = P(E_3E_4) = P(E_2E_3E_4) = \frac{1}{4}$.

- (a) $P(E_2E_3) = \frac{1}{4} = P(E_2)P(E_3)$ and hence E_2 and E_3 are independent events.
 (b) Similarly, $P(E_2E_4) = P(E_2)P(E_4)$ and $P(E_3E_4) = P(E_3)P(E_4)$, and hence E_2 , E_3 , and E_4 are pairwise independent events.
 (c) $P(E_2E_3E_4) = \frac{1}{4} \neq P(E_2)P(E_3)P(E_4) = \frac{1}{8}$ and hence E_2 , E_3 , and E_4 are *not* mutually independent events

2. (a) For what value(s) of α is $f(u) = \begin{cases} (1 + \alpha) - 2\alpha u, & 0 \leq u \leq 1, \\ 0, & \text{otherwise,} \end{cases}$
 a valid probability density function?

Solution: $\int_{-\infty}^{\infty} f(u) du = \int_0^1 (1 + \alpha) - 2\alpha u du = (1 + \alpha)u - \alpha u^2 \Big|_0^1 = 1$ for all values of α . But, for $0 \leq u \leq 1$, the graph of $f(u)$ is a straight line with endpoints $(0, 1 + \alpha)$ and $(1, 1 - \alpha)$, and does not lie above the horizontal axis unless $|\alpha| \leq 1$. Therefore, $f(u)$ is a valid pdf only when $|\alpha| \leq 1$.

- (b) Let $\mathcal{X}$ denote a continuous random variable with probability density function $f_{\mathcal{X}}(u) = \begin{cases} a + bu, & 0 \leq u \leq 1, \\ 0, & \text{otherwise.} \end{cases}$ Given that $E[\mathcal{X}] = \frac{7}{12}$, what is $P\left\{\mathcal{X} < \frac{1}{2}\right\}$?

Solution: $\int_{-\infty}^{\infty} f_{\mathcal{X}}(u) du = \int_0^1 a + bu du = au + b\frac{u^2}{2} \Big|_0^1 = \frac{2a + b}{2} = 1$ giving us that $2a + b = 2$.

Next, note that $E[\mathcal{X}] = \frac{7}{12} = \int_0^1 u(a + bu) du = a\frac{u^2}{2} + b\frac{u^3}{3} \Big|_0^1 = \frac{3a + 2b}{6}$ giving us that $6a + 4b = 7$. Solving these two equations for a and b , we get that $a = \frac{1}{2}$,

$b = 1$. Hence, $P\left\{\mathcal{X} < \frac{1}{2}\right\} = \int_0^{\frac{1}{2}} \frac{1}{2} + u du = \frac{u}{2} + \frac{u^2}{2} \Big|_0^{\frac{1}{2}} = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$.

3. Let $\mathcal{X}$ denote a Gaussian random variable with mean 10 and variance 100.

Find the value of $P\{\mathcal{X} > 31.7 | \mathcal{X} > 29.6\}$. A table of values of $\Phi(x)$, the standard Gaussian CDF, is attached to this examination.

Solution:
$$P\{\mathcal{X} > 31.7 | \mathcal{X} > 29.6\} = \frac{P(\{\mathcal{X} > 31.7\} \cap \{\mathcal{X} > 29.6\})}{P\{\mathcal{X} > 29.6\}} = \frac{P\{\mathcal{X} > 31.7\}}{P\{\mathcal{X} > 29.6\}}$$

$$= \frac{1 - F_{\mathcal{X}}(31.7)}{1 - F_{\mathcal{X}}(29.6)} = \frac{1 - \Phi\left(\frac{31.7-10}{10}\right)}{1 - \Phi\left(\frac{29.6-10}{10}\right)} = \frac{1 - \Phi(2.17)}{1 - \Phi(1.96)} = \frac{1 - 0.9850\dots}{1 - 0.9750\dots} = \frac{0.0150\dots}{0.0250\dots}$$

$$\approx 0.600\dots$$

4. Let $\mathcal{X}$ denote an exponential random variable with parameter $\lambda = 1$. Find the probability density function $f_{\mathcal{Y}}(v)$ of the random variable $\mathcal{Y} = \sqrt{\mathcal{X}}$.

To obtain full credit, you must specify the value of $f_{\mathcal{Y}}(v)$ for all v , $-\infty < v < \infty$.

Solution: Since $\mathcal{X} \geq 0$, $\mathcal{Y} \geq 0$ also. Hence, $F_{\mathcal{Y}}(v) = 0$ for $v < 0$.

For $v \geq 0$, $F_{\mathcal{Y}}(v) = P\{\mathcal{Y} \leq v\} = P\{\sqrt{\mathcal{X}} \leq v\} = P\{0 \leq \mathcal{X} \leq v^2\} = \int_0^{v^2} \exp(-u) du$

$$= 1 - \exp(-v^2). \text{ Hence, } f_{\mathcal{Y}}(v) = \begin{cases} 2v \exp(-v^2), & 0 \leq v < \infty, \\ 0, & \text{otherwise.} \end{cases}$$

This is a *Rayleigh* pdf whose general form is $\frac{v}{\sigma^2} \exp(-v^2/2\sigma^2)$ for $v \geq 0$. Rayleigh random variables are models for lifetimes of systems whose hazard rates increase linearly with time, and the amplitude of signals received over wireless channels subject to *fading*.