University of Illinois

ECE 313 **FALL 2007**

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FINAL

8:00 a.m. - 11:00 a.m., MEB 135

NOTE: This is a **closed-book closed-notes** (andclosed-neighbor) exam,with only two sheets of notes ($8\frac{1}{2}'' \times 11''$) allowed. Also, no calculators, laptops, palm pilots, etc. are allowed.

Name (Last, First) : ....

Section : ◯ Section C, 10 MWF ◯ Section D, 11 MWF

Problem 1 : .... (20 points)

Problem 2 : .... (24 points)

Problem 3 : .... (45 points)

Problem 4 : .... (36 points)

Problem 5 : .... (35 points)

TOTAL : .... (160 points)

Problem 1 (20 points)

Let X and Y be two independent jointly continuous random variables, and W and Z be two uncorrelated jointly continuous random variables. Further let α and β be two real numbers.

Read each line below carefully, and check the corresponding True box if the statement is true **in general**; otherwise check the corresponding False box. Each correct choice counts +2 points, whereas an incorrect choice counts −1 point; so guess at your own risk. You can use the space provided at the bottom as well as the facing page for scratch work.

TRUE	FALSE	
☐	☐	$\mathrm{cov}(X, 2Y) = 0$
☐	☐	$\mathrm{cov}(2W, 3Z) = 0$
☐	☐	$\mathrm{cov}(X + Y, X - Y) = 0$
☐	☐	$E[WZ] = E[W]\,E[Z]$
☐	☐	$E[W\|Z] = E[W]$
☐	☐	$E[X\|Y] = E[X]$
☐	☐	$P(W > \alpha, Z > \beta) = P(W > \alpha)\,P(Z > \beta)$
☐	☐	$\mathrm{cov}(2W, Z - W) = -2\mathrm{var}(W)$
☐	☐	$P(X > \alpha, Y < \alpha) = P(X > \alpha)\,P(Y < \alpha)$
☐	☐	$E[X^2\|Y^2] = \big(E[X\|Y]\big)^2$

END OF PROBLEM 1

Problem 2 (24 points)

Read each statement below carefully, and circle TRUE if the statement is true without any ambiguity, and circle FALSE otherwise. Each correct choice counts +3 points, whereas an incorrect choice counts −1 point; so guess at your own risk. You can use the facing page for scratch work (if needed).

(i) If X and Y are jointly continuous random variables with joint *pdf* $f_{X,Y}(x, y)$ *nonzero* and *uniform* inside the region bounded by the lines $x = \pm 1$ and $y = \pm 1$, and zero elsewhere, then X and Y are independent.

TRUE FALSE

(ii) Let X be a uniform random variable on $[-1, 1]$, and $Y = g(X)$ where $g(x) = x$ if $x \geq 0$ and zero otherwise. Then, Y is **not** a continuous random variable.

TRUE FALSE

(iii) Let X be a uniform random variable on $[-1, 1]$, and $Z = h(X)$ where $h(x) = |x|$ for all x. Then, Z is **not** a continuous random variable.

TRUE FALSE

(iv) If X, Y, Z are independent random variables, then the new random variables U, V, W defined by $U = X^2$, $V = 2Y + 3$, and $W = Z^2/(2Z + 3)$ are also independent.

TRUE FALSE

(v) If X and Y are two jointly Gaussian random variables with covariance matrix $R = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, then the correlation coefficient of X and Y is $1/2$.

TRUE FALSE

(vi) If X and Y are two independent random variables with $\text{var}(X) = 25$ and $\text{var}(Y) = 9$, then $\text{var}(X - Y) = 16$.

TRUE FALSE

(vii) X is a random variable with $\text{var}(X) = 25$ and Y is another random variable with $\text{var}(Y) = 9$. You are further told that $Y = kX$ for some real number k. Then, $\text{var}(X - Y) = 4$.

TRUE FALSE

(viii) X is a random variable with $\text{var}(X) = 25$ and Y is another random variable with $\text{var}(Y) = 9$. You are further told that $\text{var}(X - 2Y) = 61$. Then, X and Y are uncorrelated.

TRUE FALSE

END OF PROBLEM 2

Problem 3 (45 points)

Let X and Y be two jointly continuous random variables with *pdf*

$$f_{X,Y}(x,y) = \begin{cases} \text{C} & 0 < x + y \leq 1, 0 < x - y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

where C is a positive constant.

Sketch the region where $f_{X,Y}(x,y)$ is nonzero. (5 points)

Now, in each question given below and on the next page, only one of the given four answers is correct. In each case, **circle** the *letter* corresponding to the correct answer. Here each correct choice counts +4 points, whereas an incorrect choice receives −1 point. You can use the space provided at the bottom of each page as well as the facing pages for scratch work.

(i) The value of C in the expression for $f_{X,Y}(x,y)$ is equal to:

a. 1 **b.** 2 **c.** 4 **d.** none of these

(ii) The expected value of X, $E[X]$, is

a. 0 **b.** 0.5 **c.** 1 **d.** none of these

(iii) The probability $P(X \leq 0.5, Y \leq 0.25)$ is given as:

a. 7/16 **b.** 1/16 **c.** 3/16 **d.** none of these

(iv) The correlation coefficient of X, Y is given as:

a. 0 **b.** 1 **c.** -0.5 **d.** none of these

(v) The conditional expectation of X given Y, $E[X|Y]$, is given as:

a. 1 **b.** 0.5 **c.** Y **d.** none of these

(vi) The random variables X and Y are

a. uncorrelated but dependent **b.** independent **c.** positively correlated **d.** none of these

(vii) Consider the random variables $Z = X + Y$ and $W = X - Y$. The random variables Z and W are

a. uncorrelated but dependent **b.** independent **c.** positively correlated **d.** none of these

(viii) The variance of $E[X|Y]$ is

a. 0 **b.** 1 **c.** 0.5 **d.** none of these

(ix) Consider the random variable $U = X + Y - 1$. The probability $P(U > X)$ is

a. 0 **b.** 0.5 **c.** 1 **d.** none of these

(x) Consider the random variable $U = X + Y - 1$. The variance of U is

a. 1 **b.** 0.5 **c.** 1/12 **d.** none of these

END OF PROBLEM 3

Problem 4 $(3+4+5+8+8+7=36$ points)

A sample of a speech signal X is modeled as a Gaussian random variable with zero mean and unit variance. You only observe a coarsely quantized version of it:

$$Y = \operatorname{sgn}(X) = \begin{cases} -1, & X < 0 \\ 1, & X \geq 0 \end{cases}$$

(i) Write down the probability mass function (*pmf*) of Y.

(ii) Find $E[Y]$ and $\operatorname{var}(Y)$.

(iii) Determine $\operatorname{cov}(X, Y)$.

(iv) Determine the LMMSE (linear minimum mean squared error) estimator of X given Y, and the corresponding MSE (mean squared error).

(v) Write down the conditional probability density function (*pdf*) of X given $Y = 1$, that is $f_{X|Y}(x|y = 1)$. Also write down the conditional *pdf* of X given $Y = -1$, that is $f_{X|Y}(x|y = -1)$.

(vi) Determine the MMSE (minimum mean squared error) estimator of X given Y.

END OF PROBLEM 4

Problem 5 (6 + 7 + 7 + 7 + 8 = 35 points)

Consider a wireless communication system with a single transmit antenna and two receive antennas. The channel is described by

$$Y_1 = X + W_1 \qquad \text{and} \qquad Y_2 = 2X + W_2\,,$$

where X is the transmitted symbol; Y_1 and Y_2 are the received symbols at the first and the second antennas respectively; and W_1 and W_2 are *i.i.d.* Gaussian random variables with zero mean and unit variance, which are also independent of X.

(i) Find the joint probability density function (*pdf*) of Y_1 and Y_2 given $X = x$, that is $f_{(Y_1,Y_2)|X}(y_1, y_2|x)$.

(ii) Suppose $X = \pm 1$ with equal probabilities 1/2. Find the joint *pdf* of Y_1 and Y_2. that is $f_{Y_1,Y_2}(y_1, y_2)$. .

(iii) Now we want to find an estimate $\hat{X}$ for X using a linear combination of Y_1 and Y_2, that is $\hat{X} = b_1Y_1 + b_2Y_2$ where b_1 and b_2 are two positive parameters. Write down the conditional *pdf* of $\hat{X}$ given $X = 1$. Also obtain the conditional *pdf* of $\hat{X}$ given $X = -1$.

(iv) Our decision rule in concluding which symbol was sent by looking at the value of $\hat{X}$ is:

If $\hat{X} \geq 0$ then $X = 1$ was sent, and if $\hat{X} < 0$ then $X = -1$ was sent.

Compute the average probability of error under this decision rule.

Hint: You can leave the final answer in terms of the Gaussian error function.

(v) Let the decision rule be picked as in part (iv) above, and further let b_1 be fixed as $b_1 = 1$ and b_2 be a positive parameter whose value is chosen so that the average probability of error is minimized.
Find the optimal value of b_2.

END OF PROBLEM 5

SCRATCH SHEET I

SCRATCH SHEET II