SOLUTIONS TO Midterm II — ECE 313 – FALL 2007

1. (i) False ($P(X=0)=0$). (ii) True. (iii) True.

(iv) False (X is a mixed random variable). (v) True. (vi) True. (vii) False (It is equal to $\frac{1}{2}$).

(viii) False (X is a mixed random variable). (ix) True. (x) True.

2. (i) c. (ii) d. (iii) c. (iv) b. (v) c.

(vi) a. (vii) a. (viii) d. (ix) b. (x) b.

3. (i) $P(Y>\theta|X=-1) = 1-\Phi\Big(\frac{\theta-\mu+1}{\sigma}\Big)$

(ii) $P(Y<\theta|X=1) = \Phi\Big(\frac{\theta-\mu-1}{\sigma}\Big)$

(iii)

$$\begin{aligned} P_e &= pP(Y>\theta|X=-1)+(1-p)P(Y<\theta|X=1) \\ &= p\Big[1-\Phi\Big(\frac{\theta-\mu+1}{\sigma}\Big)\Big]+(1-p)\Phi\Big(\frac{\theta-\mu-1}{\sigma}\Big) \end{aligned}$$

(iv) The best decision rule is the MAP decision rule, which picks $X=1$ if $Y=y$ is such that $f_{Y|X=1}(y)/f_{Y|X=-1}(y) > p/(1-p)$ and picks $X=-1$ otherwise. The optimum value of θ is that value of y for which the inequality is an equality. Thus,

$$\begin{aligned} p\,f_{Y|X=-1}(\theta) &= (1-p)\,f_{Y|X=1}(\theta) \\ p\,e^{-\frac{\theta-\mu+1}{2\sigma^2}} &= (1-p)\,e^{-\frac{\theta-\mu-1}{2\sigma^2}} \\ \theta &= \mu+\frac{\sigma^2}{2}\ln\frac{p}{1-p} \end{aligned}$$

The result can also be obtained by differentiating the expression in (iii) with respect to θ, setting it equal to zero, and solving for θ.

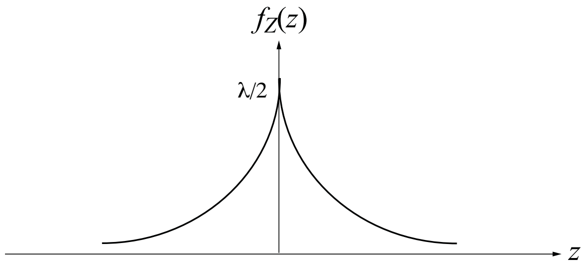
4. (i)

$$\begin{aligned} F_Z(c) &= P(Z\le c|W=1)P(W=1)+P(Z\le c|W=0)P(W=0) \\ &= \frac{1}{2}P(X\le c)+\frac{1}{2}P(-Y\le c) \\ &= \begin{cases} \frac{1}{2}P(Y\ge -c), & c<0 \\ \frac{1}{2}P(X\le c)+\frac{1}{2}, & c\ge 0 \end{cases} \\ &= \begin{cases} \frac{1}{2}e^{\lambda c}, & c<0 \\ 1-\frac{1}{2}e^{-\lambda c}, & c\ge 0 \end{cases} \end{aligned}$$

which is the cdf for a general λ. Differentiating this separately for $c>0$ and $c<0$, and letting $c=z$, leads to the pdf:

$$f_Z(z) = \frac{\lambda}{2}e^{-\lambda|z|} \qquad -\infty<z<\infty$$

(ii)



(iii) (a) $P(V=0) = P(g(Z)=0) = P(Z<-1 \text{ or } Z>3) = \frac{1}{2}e^{-\lambda} + \frac{1}{2}e^{-3\lambda}$

(iii) (b) For $0 < a < 1$,

$$\begin{aligned} P(V \le a) &= P(g(Z) \le v) \\ &= P\big(Z \le -(1-a)\big) + P\big(Z \ge 3(1-a)\big) \\ &= F_Z\big(-(1-a)\big) - F_Z\big(3(1-a)\big) \\ &= \frac{1}{2}e^{-\lambda(1-a)} - \frac{1}{2}e^{-3\lambda(1-a)} \end{aligned}$$

Therefore, we have

$$F_V(a) = \begin{cases} 0\,, & a < 0 \\ \frac{1}{2}e^{-\lambda(1-a)} - \frac{1}{2}e^{-3\lambda(1-a)}, & 0 \le a \le 1 \\ 1\,, & a > 1 \end{cases}$$

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