University of Illinois

ECE 313 **FALL 2007**

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TEST II

7:00 p.m. - 8:30 p.m.

NOTE: This is a **closed-book closed-notes** (and closed-neighbor) exam, with only one sheet of notes ($8\frac{1}{2}'' \times 11''$) allowed. Also, no calculators, laptops, palm pilots, and the like are allowed.

Name (Last, First) : ....

Section : ◯ Section C, 10 MWF ◯ Section D, 11 MWF

Problem 1 : .... (20 points)

Problem 2 : .... (40 points)

Problem 3 : .... (20 points)

Problem 4 : .... (20 points)

TOTAL : .... (100 points)

Problem 1 (20 points)

X is a random variable with cumulative distribution function (*cdf*)

$$F_X(a) = \begin{cases} 0 & a < -1 \\ (a+1)/4 & -1 \leq a < 1 \\ 1 & 1 \leq a \end{cases}$$

(Sketching $F_X(a)$ may help you in answering the questions below.)

Each of the following statements pertain to the random varaioble X whose *cdf* is given above. Read each one carefully, and check the corresponding True box if the statement is true; otherwise check the corresponding False box. Each correct choice counts +2 points, whereas an incorrect choice counts −1 point; so guess at your own risk.
You can use the space provided at the bottom as well as the facing page for scratch work.

TRUE	FALSE	
☐	☐	$P(X=0) = \frac{1}{4}$
☐	☐	$P(X=1) = \frac{1}{2}$
☐	☐	$P(\lvert X\rvert < \frac{1}{2}) = \frac{1}{4}$
☐	☐	X admits a *pdf*, which is given by $f_X(x) = \begin{cases} \frac{1}{4} & -1 < x < 1 \\ 0 & \text{else} \end{cases}$
☐	☐	The mean value of X is $E[X] = \frac{1}{2}$
☐	☐	$P(\lvert X\rvert < \frac{1}{2} \mid \lvert X\rvert \leq 1) = \frac{1}{4}$
☐	☐	$P(\lvert X\rvert < \frac{1}{2} \mid \lvert X\rvert < 1) \leq \frac{1}{4}$
☐	☐	X is a continuous random variable.
☐	☐	If $Y = 6X - 2$, the mean value of Y is $E[Y] = 1$
☐	☐	There is no value of c for which $P(X \leq c) = \frac{3}{4}$

END OF PROBLEM 1

Problem 2 (40 points)

In each question given below and on the next page, only one of the given four answers is correct. In each case, **circle** the *letter* corresponding to the correct answer. Here each correct choice counts +4 points, whereas an incorrect choice receives −1 point. There is a total of 10 questions, 5 on this page and another 5 on the next. You can use the space provided at the bottom of each page as well as the facing pages for scratch work.

(i) Only one of the following four functions is a valid *cdf*. Which one?

a. $F(b) = \begin{cases} 0 & b < -1 \\ b & -1 \leq b < 1 \\ 1 & b \geq 1 \end{cases}$ **b.** $F(b) = \begin{cases} 1 & b < -1 \\ (1-b)/2 & -1 \leq b < 1 \\ 0 & b \geq 1 \end{cases}$

c. $F(b) = \begin{cases} 0 & b < -1 \\ 1+b & -1 \leq b < -\frac{1}{2} \\ \frac{3}{4} & -\frac{1}{2} \leq b < 1 \\ 1 & b \geq 1 \end{cases}$ **d.** $F(b) = \begin{cases} 0 & b < -1 \\ 1+b & -1 \leq b < -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} \leq b < 1 \\ 1 & b \geq 1 \end{cases}$

(ii) Only one of the following four functions is a valid *pdf* of a continuous random variable. Which one?

a. $f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & \text{else} \end{cases}$ **b.** $f(x) = \begin{cases} \frac{1}{4} & 0 < x < 2 \\ \frac{1}{8}(x-2) + \frac{1}{4} & 2 \leq x < 4 \\ 0 & \text{else} \end{cases}$

c. $f(x) = \begin{cases} -1 & -1 < x < 0 \\ 2 & 0 \leq x < 1 \\ 0 & \text{else} \end{cases}$ **d.** $f(x) = \begin{cases} \frac{1}{2} & -2 < x < -1 \\ \frac{1}{4} & 1 < x < 3 \\ 0 & \text{else} \end{cases}$

(iii) Let X be a uniform random variable on the interval $[-2, 4]$ The variance of X, $\text{var}(X)$, is:

a. 0 **b.** 2 **c.** 3 **d.** 4

(iv) Let X be a Gaussian random variable with mean value $\mu = 2$, and variance $\sigma^2 = 4$, that is $X \sim N(2, 4)$. If Φ denotes the *cdf* of the standard Gaussian random variable, $P(-1 < X \leq 2)$ is:

a. $\Phi(\frac{1}{4}) - \Phi(-\frac{1}{4})$ **b.** $\Phi(\frac{3}{2}) - \frac{1}{2}$ **c.** $\Phi(2) - \Phi(-1)$ **d.** none of these

(v) Let $X \sim N(2, 4)$ as in (iv) above, and $Y = 3X^2 + 2X$. The mean value of Y is:

a. 12 **b.** 16 **c.** 28 **d.** none of these

(vi) Let X be a gamma random variable with parameter $\lambda = 2$ and order $\alpha = 4$. The variance of X, var(X), is:

a. 1 **b.** 0.5 **c.** 2 **d.** 4

(vii) Let X be a uniform random variable on the interval $[-2, 4]$. Let $Y = e^{2X}$. The *pdf* of Y is :

a. $f_Y(y) = \begin{cases} 1/(12y) & e^{-4} \le y \le e^8 \\ 0 & \text{else} \end{cases}$ **b.** $f_Y(y) = \begin{cases} 1/(2y) & e^{-4} \le y \le e^8 \\ 0 & \text{else} \end{cases}$

c. Y has a *pdf*, but it is none of these two **d.** Y does not have a *pdf*

(viii) The hazard rate function of a positive random variable X is given as $\lambda(t) = 3t\,,\ t \ge 0$. The *pdf* of X, $f_X(x)$, is for all $x \ge 0$:

a. $1 - e^{-x^2}$ **b.** $3xe^{-x^2}$ **c.** $3e^{-3x}$ **d.** none of these

(ix) Two discrete random variables, X and Y, take on only four values each, -1, 0, 1, and 2. The joint probability mass function (*pmf*) of the pair (X, Y) is $p_{X,Y}(i, j) = c$ for all $i, j = -1, 0, 1, 2$. The value of the constant c is:

a. $\frac{1}{4}$ **b.** $\frac{1}{16}$ **c.** $\frac{1}{8}$ **d.** none of these

(x) Let X and Y be as in (ix) above. The expected value of Y, $E[Y]$, is:

a. 1 **b.** $\frac{1}{2}$ **c.** 2 **d.** none of these

END OF PROBLEM 2

Problem 3 $(4 + 4 + 5 + 7 = 20$ points)

Let X and Z be two independent random variables, where Z is a Gaussian random variable with mean μ and variance σ^2, and X is a discrete random variable taking two values: -1 with probability p and 1 with probability $1 - p$. Let Y be another random variable related to X and Z by:

$$Y = X + Z$$

(The answers to the four questions below will be in terms of all or some of the parameters p, μ and σ. Further, when necessary, you can express the answers in terms of the Gaussian error function (standard Gaussian *cdf*) Φ, defined by $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-(y^2/2)} dy$.)

(i) Given that $X = -1$ (that is, conditioned on $X = -1$), what is the probability that $Y > \theta$, where θ is a real number?

$$P(Y > \theta \,|\, X = -1) =$$

(ii) Given that $X = 1$ (that is, conditioned on $X = 1$), what is the probability that $Y < \theta$?

$$P(Y < \theta \,|\, X = 1) =$$

(iii) Let H_0 denote the hypothesis that $X = -1$, and H_1 denote the hypothesis that $X = 1$. We decide in favor of H_0 if $Y < \theta$ and H_1 if $Y > \theta$.
Find the average probability of error in terms of the parameters θ, p, μ and σ.

Average Prob. Error =

(iv) Find the value of θ that minimizes the average probability of error you have obtained above.

Optimum θ =

END OF PROBLEM 3

Problem 4 $(7(5+3)+2+10(3+7)=20$ points)

Let X, Y and W be three independent random variables, where X and Y are exponential random variables with the same parameter value $\lambda = 1$, and W is a Bernoulli random variable with parameter $p = \frac{1}{2}$. Let Z be another random variable, expressed in terms of X, Y and W as follows:

$$Z = WX - (1 - W)Y$$

That is, Z is X when $W = 1$, and Z is $-Y$ when $W = 0$.

(i) Obtain the expressions for the cumulative distribution function (*cdf*) and the probability density function (*pdf*) of Z.

$$F_Z(c) = \begin{cases} \\ \\ \end{cases} \qquad\qquad f_Z(z) = \begin{cases} \\ \\ \end{cases}$$

(ii) Sketch the *pdf* of Z.

(iii) Let $V = g(Z)$, where

$$g(z) = \begin{cases} z+1, & -1 \le z < 0 \\ 1 - \frac{z}{3}, & 0 \le z < 3 \\ 0, & \text{otherwise} \end{cases}$$

(a) Compute the probability of the event $\{V = 0\}$, and **(b)** obtain the expression for the *cdf* of V. (In answering these two questions, you will find it useful to sketch the function g.)

$$P(V = 0) =$$

$$F_V(a) = \begin{cases} \\ \\ \\ \end{cases}$$

END OF PROBLEM 4

SCRATCH SHEET I

SCRATCH SHEET II