SOLUTIONS TO Midterm I — ECE 313 – FALL 2007

1. True, True, False, False, True, True, True, False, False, True.

Brief reasons:

(i) $P(E^c \cup F^c) = P((EF)^c) = 1 - P(EF)$.

(ii) $1 - P(E^c|F^c)P(F^c) = 1 - P(E^cF^c) = 1 - P((E \cup F)^c) = 1 - 1 + P(E \cup F)$.

(iii) If E and F are independent, the LHS is $2P(E)$, which cannot equal the RHS for all E.

(iv) The two sides are equal only if $P(E) = P(F)$.

(v) This is total probability relationship applied to conditional probabilities.

(vi) You can easily see this using Venn diagrams.

(vii) The LHS is: $P(EF) + P(E^cF) = P(F)$.

(viii) We also need pairwise independence.

(ix) Since E and F are mutually exclusive, $P(E|F) = 0$; but $P(E) > 0$.

(x) You are given that $P(E|F) > P(E)$, which is equivalent to $P(EF) > P(E)P(F)$. From this it follows, by dividing both sides by $P(E)$, that $P(F|E) > P(F)$.

2. (i) b, (ii) b, (iii) a, (iv) c, (v) d, (vi) a, (vii) b, (viii) d, (ix) b, (x) a.

Brief reasons:

(i) Use Venn diagram to see this immediately.

(ii) $P(F|E) = P(EF)/P(E) = 0.2/0.4 = 0.5$.

(iii) $\text{var}(X) = np(1-p) = (10)(0.4)(0.6) = 2.4$.

(iv) $P(X \geq 2) = 1 - P(X = 0) - P(X = 1) = 1 - e^{-3} - 3e^{-3}$

(v) G and F are mutually exclusive, and hence given F the conditional probability of the event G is 0.

(vi) $(-1)(0.4) + (1)(0.2) + (2)(0.4) = 0.6$.

(vii) Note that $X^2 - 1$ takes on only two values: 0 with probability 0.6, and 3 with probability 0.4.

(viii) In the Chebyshev's inequality, $\sigma = 2, a = 2$. Hence, $K = 1/a^2 = 0.25$.

(ix) $0.6\hat{p} = 120/500$.

(x) The probability of both coming heads is $(0.6)(0.5) = 0.3$. We are then looking for the largest integer smaller than (0.3) (501), which is 150.

3. (i) The desired probabability, when the random variable is modeled as Poisson, is

$$1 - \sum_{i=0}^{4} \frac{e^{-2}2^i}{i!} = 1 - 7e^{-2}$$

(ii) (a) Expression for the desired probability when the random variable is modeled as Binomial, is

$$1 - \sum_{i=0}^{4} \frac{2000!}{(2000-i)!\,i!} 10^{-3i} 0.999^{2000-i}$$

(ii) (b) Yes, because Poisson is a good approximation to Binomial when n is large and p is small. Hence, the expression in (a) can be computed fairly accurately to be $1 - 7e^{-2}$.

4. (i) When $\alpha \leq 1 - \beta$, $P(\text{error}|X = 0) = \alpha \qquad P(\text{error}|X = 1) = \beta$.

When $\alpha > 1 - \beta$, $P(\text{error}|X = 0) = 1 - \alpha \qquad P(\text{error}|X = 1) = 1 - \beta$.

(ii) MAP detection rule is

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\begin{aligned}
Y = 0: \quad & \frac{1-p}{p} \underset{0}{\overset{1}{\gtrless}} 3 \;\equiv\; 0.25 \underset{0}{\overset{1}{\gtrless}} p \\
Y = 1: \quad & \frac{1-p}{p} \underset{0}{\overset{1}{\gtrless}} \frac{1}{7} \;\equiv\; 0.875 \underset{0}{\overset{1}{\gtrless}} p
\end{aligned}
$$

$$
P(\text{error}) = \begin{cases} p, & p \leq 0.25 \\ 0.3 - 0.2p, & 0.25 < p \leq 0.875 \\ 1 - p, & p > 0.875 \end{cases}
$$

(iii) We have $\tau = \frac{7}{18}$. As $\frac{\beta}{1-\alpha} < \tau$ and $\frac{1-\beta}{\alpha} > \tau$, the Bayes' minimum average cost decision rule is: when $Y = 0$, $\hat{X} = 0$, and when $Y = 1$, $\hat{X} = 1$.

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