ECE 413 **FALL 2005**

November 15, 2005

SOLUTION TO TEST II AND SOME STATISTICS

Problem 1

The correct choices are: F, F, F, T, F, T, F, T, T, T. Brief reasons:

(i) $F_X(b)$ is continuous at $b = 0$, and hence $P(X = 0) = 0$.

(ii) $F_X(b)$ has a jump at $b = 1$ of height $\frac{1}{2}$.

(iii) This is equal to $1 - P(X \le \frac{1}{2} = \frac{7}{8}$.

(iv) This is equal to $\frac{1}{2} - \frac{1}{8} = \frac{3}{8}$.

(v) Since F_X is not differentiable, *pdf* does not exist.

(vi) $\int_0^1 x^2 dx + \frac{1}{2}(1) = \frac{5}{6}$.

(vii) $\int_0^1 x^3 dx + \frac{1}{2}(1)^2 = \frac{3}{4}$.

(viii) $P(X < \frac{1}{2} = \frac{1}{8}$, $P(X < 1 = \frac{1}{2}$, and the conditional probability is the ratio of these two (since intersection of $\{X < \frac{1}{2}\}$ and $\{X < 1\}$ is $\{X < \frac{1}{2}\}$).

(ix) $E[Y] = 6E[X] - 2 = 5 - 2 = 3$.

(x) The point where F_X has discontinuity has mass $\frac{1}{2}$.

Problem 2

The correct choices are: **d, c, d, b, c, a, a, b, b, a**. Brief reasons:

(i) The first three are not nondecreasing.

(ii) First two have areas > 1, and the last one is not nonnegative.

(iii) $E[X] = 0.5$ and $E[X^2] = 1$.

(iv) $P(1 < X < 3) = P(-1 < X - 2 < 1) = P(-\frac{1}{3} < \frac{1}{3}(X - 2) < \frac{1}{3})$.

(v) $E[Y] = 2E[X^2] + E[X] = 2(4 + 9) + 2 = 28$.

(vi) $\text{var}(X) = \alpha/\lambda^2$.

(vii) $f_Y(y) = f_X((\ln y)/2)/2y$ for $e^{-2} \le y \le e^4$.

(viii) $F_X(t) = 1 - \exp\left(-\int_0^t \lambda(s)ds\right) = 1 - \exp(-t^2)$, and f_X is its derivative.

(ix) There are $3^2 = 9$ combinations of i and j.

(x) Y takes the values 0, 1, and 2, with equal probability.

Problem 3

(i) Under the ML DR, we decide on H_0 as the true hypothesis whenever $f_0(x) > f_1(x)$. Since $f_0(x) = \exp(-x)$ and $f_1(x) = 2\exp(-2x)$, for $x \ge 0$, and *zero* otherwise, the range of values for x (for H_0 to be declared to be the true hypothesis) is: $\frac{1}{2}\exp(x) > 1 \quad \Rightarrow \quad x > \ln 2$.

(ii) $p_{\text{FA}} = P(X < \ln 2 | H_0) = \int_0^{\ln 2} \exp(-x)dx = 1 - \frac{1}{2} = \frac{1}{2}$

$p_{\text{MD}} = P(X > \ln 2 | H_1) = 2\int_{\ln 2}^{\infty} \exp(-2x)dx = \frac{1}{4}$

$p_{\text{total}} = \pi_0 p_{\text{FA}} + \pi_1 p_{\text{MD}} = 0.4\frac{1}{2} + 0.6\frac{1}{4} = 0.35$

(iii) Now the threshold is π_0/π_1 instead of 1. That is, we decide H_0 over H_1 for those values of x satisfying $\pi_0 f_0(x) > \pi_1 f_1(x)$. This leads to: $\frac{2}{3}\frac{1}{2}\exp(x) > 1 \quad \Rightarrow \quad x > \ln 3$. Hence, decide H_0 as the true

hypothesis if $x > \ln 3$, and decide on H_1 if $0 \leq x < \ln 3$. For $x = \ln 3$, which occurs with probability 0, we are indifferent between H_0 and H_1. This is the MAP rule. The corresponding probability of error is $p_{\text{total}} = \pi_0 P(X < \ln 3|H_0) + \pi_1 P(X > \ln 3|H_1) = 0.4(1 - \frac{1}{3}) + 0.6\frac{1}{9} = \frac{1}{3}$ which is smaller than the p_{total} of part (ii), as to be expected.

Problem 4

(i) Simple integration in each case leads to

$$F_X(a) = \begin{cases} 0 & a < -1 \\ \frac{1}{2}(1 + a^2 + 2a) & -1 \leq a < 0 \\ \frac{1}{2}(1 - a^2 + 2a) & 0 \leq a < 1 \\ 1 & a \geq 1 \end{cases} \qquad F_Y(b) = \begin{cases} 0 & b < -1 \\ \frac{1}{3} & -1 \leq b < 0 \\ \frac{2}{3} & 0 \leq b < 1 \\ 1 & b \geq 1 \end{cases}$$

(ii) $P(-0.5 < X \leq 1) = F_X(1) - F_X(-0.5) = 1 - \frac{1}{2}(1 + 0.25 - 1) = \frac{7}{8}$
$P(-0.5 < Y \leq 1) = F_Y(1) - F_Y(-0.5) = 1 - \frac{1}{3} = \frac{2}{3}$

(iii) $P(-0.5 < Z \leq 1) = F_Z(1) - F_Z(-0.5) = \frac{1}{2}(\,F_X(1) - F_X(-0.5)\,) + \frac{1}{2}(\,F_Y(1) - F_Y(-0.5)\,) = \frac{1}{2}\left(\frac{7}{8} + \frac{2}{3}\right) = \frac{37}{48} = 0.77$

(iv) $P(-0.5 < Z \leq 1||Z| < 1) = P(-0.5 < Z < 1)/P(|Z| < 1) = \left(\frac{37}{48} - \frac{1}{6}\right)/\left(\frac{1}{2} + \frac{1}{6}\right) = \frac{29}{48}\frac{6}{4} = \frac{29}{32} = 0.906$

(v) Both X and Y are zero-mean random variables, and hence both *cdf*s F_X and F_Y have zero means. Hence, the mean value of Z is also *zero*: $E[Z] = 0$.
The second moment of X is: $E[X^2] = \int_{-1}^{0} x^2(1 + x)dx + \int_0^1 x^2(1 - x)dx = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$
and the second moment of Y is: $E[Y^2] = \frac{1}{3}(1 + 0 + 1) = \frac{2}{3}$.
Then, the second moment of Z is $E[Z^2] = \frac{1}{2}\left(\frac{1}{6} + \frac{2}{3}\right) = \frac{5}{12}$.
Since the mean value of Z is *zero*, its variance is equal to its second moment, and hence $\text{var}(Z) = \frac{5}{12}$.

STATISTICS ON TEST II

	Average	*Maximum*	*Minimum*	*Median*
Problem 1	15.74 (78.7 %)	20	02	
Problem 2	33.41 (83.5 %)	40	08	
Problem 3	15.47 (77.4 %)	20	03	
Problem 4	15.09 (75.4 %)	20	03	
TOTAL	**79.71**	99	42	**82**

COMBINED STATISTICS ON TESTS I & II

Average	*Maximum*	*Minimum*	*Median*
75.13	99	41	**76**
