ECE 413 **FALL 2005**

October 8, 2005

SOLUTION TO TEST I AND SOME STATISTICS

Problem 1

The correct choices are: **b, d, a, c, c, d, c, c, c, a**. Brief reasons:

(i) $6\binom{5}{2} = \frac{(6)5!}{2!3!} = \frac{6!}{2!3!}$.

(ii) Let T denote the two-headed coin. $P(T|H) = P(H|T)P(T)/P(H) = (1)(\frac{1}{3})/[\frac{1}{3}(1+\frac{1}{2}+\frac{1}{2})] = \frac{1}{2}$.

(iii) Follows readily from a Venn diagram.

(iv) Because E and G are mutually exclusive

(v) E is independent of F_1 and F_2, and hence also independent of $F_1 \cup F_2 \equiv F_3^c$ since F_1 and F_2 are mutually exclusive. Thus, $P(E|F_3) = P(E)$

(vi) For a Poisson random variable, the second moment is $\lambda + \lambda^2$.

(vii) Largest integer smaller than (1.5)(7)=10.5

(viii) $(-2)^3(0.3) + (0)^3(0.2) + (2)^3(0.5) = 1.6$

(ix) By Chebyshev's inequality $P(X^4 \geq 16) = P(|X| \geq 2) \leq \sigma^2/2^2 = 0.5/4 = 0.125$.

(x) It follows from (ix) above, since $P(|X| < 2) = 1 - P(|X| \geq 2)$.

Problem 2

The correct choices are: F, T, F, F, T, F, F, F, F, T. Brief reasons:

(i) E, F, G can be picked to make the sum of their probabilities larger as well as smaller than 1.

(ii) Simple application of De Morgan's rules

(iii) The RHS could be larger than 1 (pick $EFG = \emptyset$, $P(E) = P(F) = P(G) = 0.4$).

(iv) We also need $P(FE^c)$ as an additive term on the LHS.

(v) If $EFG = \emptyset$, each term is 0, and if $EFG \neq \emptyset$, each term is 1.

(vi) If $G \subset E$ and $G \subset F$, the LHS is 1 while the RHS is 2.

(vii) If $F \subset E$, the LHS is 1.

(viii) The RHS should also have -P(FG), and independence of F and G does not make this term zero.

(ix) The RHS should be P(G).

(x) Probability of the union of two events cannot be smaller than the probability of any of the individual events.

Problem 3

(i) X's *pmf* is a weighted sum of *pmf*s of two geometric random variables:

$$P(X = n) = P(X = n|C_r)P(C_r) + P(X = n|C_s)P(C_s) = (1-r)^{n-1}r.\frac{2}{5} + (1-s)^{n-1}s.\frac{3}{5}$$

$$E[X] = \sum_n n(1-r)^{n-1}r.\frac{2}{5} + \sum_n n(1-s)^{n-1}s.\frac{3}{5} = \frac{1}{5}(\frac{2}{r} + \frac{3}{s})$$

(Used the fact that for a geometric random variable with parameter r, the mean value is $1/r$.)

$$E[X^2] = \sum_n n^2(1-r)^{n-1}r.\frac{2}{5} + \sum_n n^2(1-s)^{n-1}s.\frac{3}{5} = \frac{1}{5}(\frac{4}{r^2} - \frac{2}{r} + \frac{6}{s^2} - \frac{3}{s})$$

(Used the fact that for a geometric random variable as above, the second moment is $(1/r^2)(2-r)$.)

Hence, Var $(X) = E[X^2] - \big(E[X]\big)^2 = \frac{1}{25}\left[\frac{16}{r^2} - \frac{10}{r} + \frac{21}{s^2} - \frac{15}{s} - \frac{12}{rs}\right]$

(ii) $P(Y = n) = P(Y = n|A)P(A) + P(Y = n|B)P(B) = \frac{1}{2}\left[(1-r)^{n-1}r.(\frac{2}{5}+\frac{1}{2}) + (1-s)^{n-1}s.(\frac{3}{5}+\frac{1}{2})\right]$

$$= \frac{1}{20}\left[9(1-r)^{n-1}r + 11(1-s)^{n-1}s\right]$$

Now, using Bayes' rule, $P(C_r|Y = k) = P(Y = k|C_r)P(C_r)/P(Y = k)$.

To complete the derivation, we now have to compute $P(Y = k|C_r)P(C_r)$, as we already have an expression for the denominator above.

$$\begin{aligned} P(Y = k|C_r)P(C_r) &= P(Y = k|C_r, A)P(A|C_r)P(C_r) + P(Y = k|C_r, B)P(B|C_r)P(C_r) \\ &= P(Y = k|C_r, A)P(C_r|A)P(A) + P(Y = k|C_r, B)P(C_r|B)P(B) \\ &= \frac{1}{2}.\frac{2}{5}(1-r)^{k-1}r + \frac{1}{2}.\frac{1}{2}(1-r)^{k-1}r = \frac{9}{20}(1-r)^{k-1}r \end{aligned}$$

where we have used the numerical values: $P(C_r|A) = 2/5, P(C_r|B) = 1/2$. Hence the solution is:

$$P(\text{picked of type } C_r|Y = k) = \frac{9(1-r)^{k-1}r}{9(1-r)^{n-1}r + 11(1-s)^{n-1}s}$$

(iii) We need to maximize $P(X = 2)$, over s, with $r = 2s$. From part (i), the function to be maximized is $(2/5)2s(1-2s) + (3/5)s(1-s)$. Differentiating this with respect to s, and setting the derivative equal to zero, leads to the unique solution: $\hat{s} = 7/22$. The second derivative is $(-22/5) < 0$, and hence this is indeed a maximizing solution. Since $2\hat{s} < 1$, it is a legitimate probability (for both s and r).

Problem 4

(i) $P(X = k) = P(X = k|\mu = 1)P(\mu = 1) + P(X = k|\mu = 0)P(\mu = 0) = \frac{2^k e^{-2}}{k!}p + \frac{e^{-1}}{k!}(1-p)$

(ii) $E[X] = \sum_k k\frac{2^k e^{-2}}{k!}p + \sum_k k\frac{e^{-1}}{k!}(1-p) = 2p + (1-p) = p + 1$

where we have made use of the fact that a Poisson random variable with rate λ has mean value λ.

(iii) $E[X^2] = \sum_k k^2\frac{2^k e^{-2}}{k!}p + \sum_k k^2\frac{e^{-1}}{k!}(1-p) = (2+4)p + (1+1)(1-p) = 2(2p+1)$

where we have made use of the fact that a Poisson random variable with rate λ has second moment $\lambda + \lambda^2$. Then, $\text{Var}(X) = E[X^2] - \big(E[X]\big)^2 = 2p - p^2 + 1$

(iv) $P(X > 0) = 1 - P(X = 0) = 1 - e^{-2}p - e^{-1}(1-p) = 1 - e^{-1} + p\big(e^{-1} - e^{-2}\big)$

STATISTICS ON TEST I

	Average	*Maximum*	*Minimum*	*Median*
Problem 1	25.87 (65 %)	40	06	
Problem 2	14.64 (73 %)	20	03	
Problem 3	14.62 (73 %)	20	01	
Problem 4	10.47 (52 %)	20	00	
TOTAL	**65.60**	100	18	**66**
