University of Illinois

ECE 413 **FALL 2005**

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TEST I

7:00 p.m. - 8:30 p.m.

NOTE: This is a **closed-book closed-notes** (and closed-neighbor) exam, with only one sheet of notes ($8\frac{1}{2}'' \times 11''$) allowed. Also, no calculators, laptops, palm pilots, and the like are allowed.

Name (Last, First) : ....

Section : ◯ Section C, 10 MWF ◯ Section D, 11 MWF

Problem 1 : .... ($4 \times 10 = 40$ points)

Problem 2 : .... ($2 \times 10 = 20$ points)

Problem 3 : .... ($8 + 6 + 6 = 20$ points)

Problem 4 : .... ($5 + 5 + 5 + 5 = 20$ points)

TOTAL : .... (100 points)

Problem 1 (40 points)

In each question given below and on the next page, only one of the given four answers is correct. In each case, **circle** the *letter* corresponding to the correct answer. Here each correct choice counts +4 points, whereas an incorrect choice receives −1 point. There is a total of 10 questions, 5 on this page and another 5 on the next. You can use the space provided at the bottom of each page as well as the facing pages for scratch work.

(i) A president and 2 vice-presidents (identical positions) are to be elected from a slate of 6 candidates. If no one can hold more than one office (note that there are three positions), how many different outcomes can we have as a result of the election?

a. $\frac{6!}{3!}$ **b.** $\frac{6!}{3!2!}$ **c.** $\frac{6!}{2!}$ **d.** none of these

(ii) An urn contains two fair coins and one two-headed coin. A coin is selected at random and tossed once. If a head is observed, what is the probability that the coin selected was the two-headed one?

a. 2/3 **b.** 3/4 **c.** 1/3 **d.** none of these

(iii) Let E, F, G be three events defined on a common sample space, with the properties: E and G are mutually exclusive, and $P(EF) = 0.1$, $P(FG) = 0.2$, $P(E) = 0.5$, $P(G) = 0.4$, $P(E \cup F \cup G) = 1$. Compute $P(F)$.

a. 0.4 **b.** 0.6 **c.** 0.2 **d.** none of these

(iv) In (iii) above, $P(E|FG)$, the conditional probability of E given F and G, is

a. 0.25 **b.** 0.5 **c.** 0 **d.** none of these

(v) On a sample space Ω, you are given three mutually exclusive events, F_1, F_2, F_3, with the further property that $F_1 \cup F_2 \cup F_3 = \Omega$. Let E be another event on the same sample space, with probability $P(E) = 1/2$. You are also given the conditional probabilities $P(E|F_1) = P(E|F_2) = 1/2$.
What is the conditional probability of E, given F_3, that is $P(E|F_3)$?

a. 1/6 **b.** 1/4 **c.** 1/2 **d.** none of these

(vi) Let X be a Poisson random variable with parameter $\lambda = 3$. The second moment of X, that is $E[X^2]$, is

a. 3 **b.** 9 **c.** 6 **d.** 12

(vii) Let Y denote the number of cars that cross a particular intersection per minute during the morning hours on Sundays, modeled as a Poisson random variable with parameter $\lambda = 1.5$. Let Z denote the random variable which denotes the total number of cars that cross the intersection in a 7-minute period. Find the integer k that maximizes $\text{Prob}\,(Z = k)$.

a. 1 **b.** 2 **c.** 10 **d.** 11

(viii) Let X be a discrete random variable with probability mass function (pmf)

$$P(X = i) = \begin{cases} 0.3 & \text{for } i = -2 \\ 0.2 & \text{for } i = 0 \\ 0.5 & \text{for } i = 2 \end{cases}$$

The third moment of X, $E[X^3]$, is

a. 6.4 **b.** 1 **c.** 1.6 **d.** none of these

(ix) X is a zero-mean discrete random variable with variance 0.5. Use Chebyshev's inequality to obtain the smallest value of the real number K such that $P(X^4 \geq 16) \leq K$.

a. $K = 0.875$ **b.** $K = 0.5$ **c.** $K = 0.125$ **d.** $K = 0.25$

(x) X is a zero-mean discrete random variable with variance 0.5. Use Chebyshev's inequality to obtain the largest value of the real number k such that $P(|X| < 2) \geq k$.

a. $k = 0.875$ **b.** $k = 0.5$ **c.** $k = 0.125$ **d.** $k = 0.75$

END OF PROBLEM 1

Problem 2 (20 points)

Let E, F, and G be three events defined on a common sample space, with $0 < P(E) < 1$, $0 < P(F) < 1$, and $0 < P(G) < 1$. (*Note the strict inequality in each case.*)

Read each of the following statements carefully, and check the corresponding True box if the statement is always true (that is, it holds for all E, F, G); otherwise check the corresponding False box. Each correct choice counts +2 points, whereas an incorrect choice counts −1 point; so guess at your own risk.
You can use the space provided at the bottom as well as back of the previous page for scratch work.

TRUE	FALSE	
☐	☐	$P(E) + P(F) + P(G) = 1$
☐	☐	$P(E[F^c \cup G^c]^c \cup G) = P(G)$
☐	☐	$P(E \cup F \cup G) \geq P(E) + P(F) + P(G) - P(EFG)$
☐	☐	$P(EF) + P(E^cF^c) + P(EF^c) = 1$
☐	☐	$P(E\|EFG) = P(F\|EFG) = P(G\|EFG)$
☐	☐	$P(E \cup F\|G) = P(E\|G) + P(F\|G)$
☐	☐	$P(E\|F) \leq P(E)$
☐	☐	If $P(F\|G) = P(F)$, then $P(F \cup G) = P(F) + P(G)$
☐	☐	$P(F\|G)P(G) + P(F^c\|G)P(G) = P(F)$
☐	☐	$P(E \cup F) \geq \max\{P(E), P(F)\}$

END OF PROBLEM 2

Problem 3 (8+6+6 = 20 points)

We are given a bag, A, that contains two types of biased coins, denoted C_r and C_s. The probability of observing a Head is r for a coin of type C_r, and s for a coin of type C_s. Both r and s are positive. In the bag, there are 10 coins of type C_r and 15 of type C_s.
(In answering the three questions below, you must show details of your derivation and reasoning. The answers will be in terms of the parameters r and s.)

(i) Suppose we randomly pick a coin from the bag, toss it repeatedly, and stop as soon as we see a Head. Let X denote the number of times we toss the coin before we stop.

Obtain the **pmf** of X, its **mean** value, and its **variance**.

$P(X = n) =$
for $n = 1, 2, \ldots$

$E[X] =$

$\text{Var}(X) =$

(ii) Now suppose that we are given another bag, B, which contains 10 coins of type C_r and 10 coins of type C_s. We first randomly pick one of the two bags, A or B, and then randomly pick a coin from this bag, toss it repeatedly, and stop the first time we see a Head. Let Y denote the number of times we toss the coin until we see a Head.

(a) First obtain an expression for the **pmf** of Y.

(b) Next suppose that Y has been observed to have the specific value k, where k is an integer. Given this observation, what is the probability that we were tossing a coin of type C_r?

(The answer in (b) should obviously also depend on k. You can express the conditional probability here in terms of the pmf of Y which you obtained in (a).)

(a) $P(Y = n) =$

for $n = 1, 2, \ldots$

(b) $P(\text{we picked a coin of type } C_r | Y = k) =$

(iii) Let us again consider the scenario in part (i) (with a single bag A). Suppose that you do not know the true values of r and s, but you know that $r = 2s$. Further, you observe Head (for the first time) in the second toss. What is the maximum likelihood estimate (**MLE**) for s?

$\hat{s} =$

END OF PROBLEM 3

Problem 4 (5+5+5+5 = 20 points)

(*You should show your steps in arriving at the final answers.*)

Let μ be a Bernoulli random variable with parameter p, that is $P(\mu = 1) = p$. Let X be a discrete random variable taking nonnegative integer values $\{0, 1, 2, \ldots\}$, with the property that the conditional probability of the event $\{X = k\}$ given the event $\{\mu = i\}$ is

$$P(X = k|\mu = i) = \frac{(i+1)^k e^{-(i+1)}}{k!}\,, \quad k = 0, 1, 2, \ldots, \quad \text{and} \quad i = 0, 1\,.$$

(i) Obtain the probability mass function **(pmf)** of X (as a function of the parameter p).

$P(X = k) =$

for $k = 0, 1, \ldots$

(ii) What is the **expected value** of X (again as a function of the parameter p)?

$E[X] =$

(iii) Obtain the **variance** of X (again as a function of the parameter p).

$\mathrm{Var}(X) =$

(iv) Compute the probability that X is **positive** (again as a function of the parameter p).

$P(X > 0) =$

END OF PROBLEM 4 / END OF TEST

SCRATCH SHEET