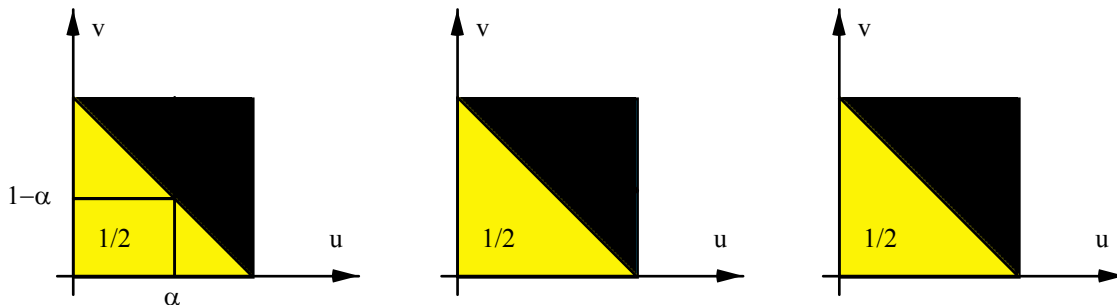


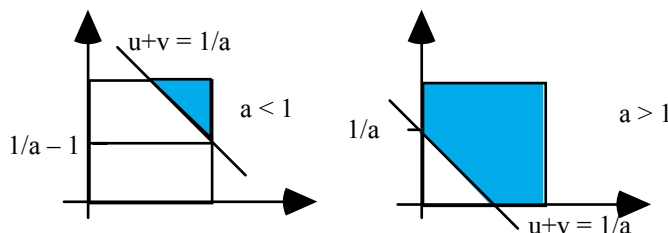
78.



The joint pdf has values as shown in the two triangular regions.

- (a) $f_{\mathbf{X}}(\alpha) = \text{area of cross-section at } \alpha = \text{sum of areas of two rectangles} = (1/2)(1 - \alpha) + (3/2)\alpha = (1/2) + \alpha$, for $0 \leq \alpha \leq 1$, and 0 if $\alpha < 0$ or $\alpha > 1$.
- (b) $P\{\mathbf{X} + \mathbf{Y} \leq 3/2\} = 1 - P\{\mathbf{X} + \mathbf{Y} > 3/2\} = 1 - \text{volume above heavy line in middle figure above}$
 $= 1 - (1/2) \times (1/2) \times (1/2) \times (3/2) = 13/16$.
- (c) $P\{\mathbf{X}^2 + \mathbf{Y}^2 \geq 1\} = \text{Volume outside heavy curve in right-hand figure above} = (1 - \pi/4) \times (3/2) = 3/2 - 3\pi/8$.
 (Alternatively, volume inside the curve is $(1/2)^2 + (\pi/4 - 1/2) \times 3/2 = 3\pi/8 - 1/2$)

79. $\mathbf{I} = 1/(\mathbf{R}_1 + \mathbf{R}_2)$ takes on values in the range $(1/2, \infty)$. Now, $F_{\mathbf{I}}(a) = P\{\mathbf{I} \leq a\} = P\{1/(\mathbf{R}_1 + \mathbf{R}_2) \leq a\}$
 $= P\{\mathbf{R}_1 + \mathbf{R}_2 \leq a^{-1}\}$. Now, $1 < a < \infty$ implies that $a^{-1} < 1$ while $1/2 < a < 1$ implies that $1 < a^{-1} < 2$. Thus,
 we have the two cases to consider as illustrated in the figure below.



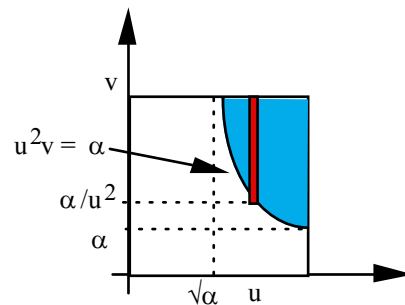
It follows readily from the diagrams that

$$F_{\mathbf{I}}(a) = \begin{cases} 0, & a < 1/2, \\ (2 - a^{-1})^2/2, & 1/2 \leq a \leq 1, \\ 1 - a^{-2}/2, & 1 < a < \infty, \end{cases} \quad f_{\mathbf{I}}(a) = \begin{cases} 2a^{-2} - a^{-3}, & 1/2 \leq a \leq 1, \\ a^{-3}, & 1 < a < \infty, \end{cases}$$

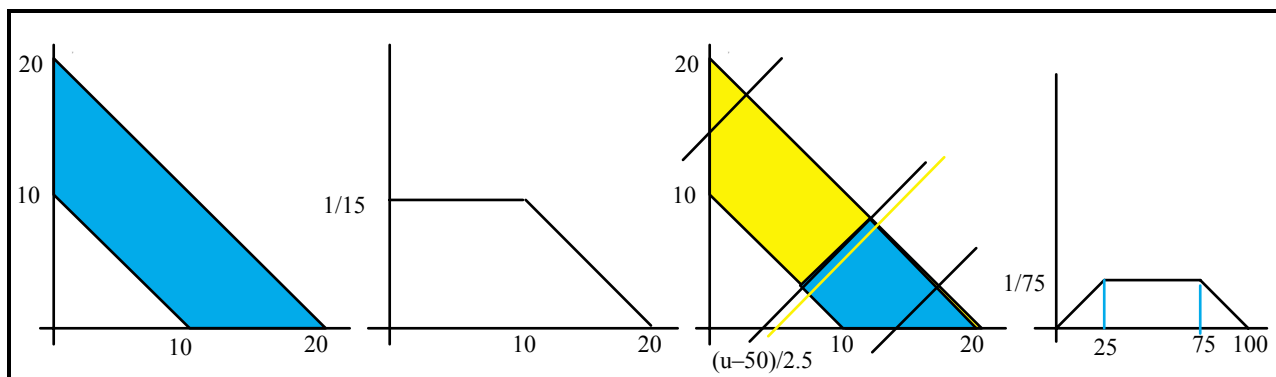
80. $\mathbf{Z} = \mathbf{X}^2\mathbf{Y}$. Then, $0 < \mathbf{Z} < 1$, and for $0 < \alpha < 1$,

$$\begin{aligned} P\{\mathbf{Z} > \alpha\} &= P\{\mathbf{X}^2\mathbf{Y} > \alpha\} \\ &= \int_{u=\sqrt{\alpha}}^1 \int_{v=\alpha/u^2}^1 2u \, dv \, du = \int_{u=\sqrt{\alpha}}^1 2u(1 - \alpha/u^2) \, du \\ &= u^2 - 2\alpha \ln u \Big|_{\sqrt{\alpha}}^1 = (1 - \alpha) - (2\alpha \ln 1 - 2\alpha \ln \sqrt{\alpha}). \end{aligned}$$

Hence, $1 - F_{\mathbf{Z}}(\alpha) = 1 - \alpha + \alpha \ln \alpha$ for $0 < \alpha < 1$, and the pdf $f_{\mathbf{Z}}(\alpha)$ is given by $-\ln \alpha$, $0 < \alpha < 1$, and 0 elsewhere.



81.(a) The joint pdf has constant value K on the shaded region shown in the left-hand figure below. The area can be computed as the difference in area of two triangles as $(1/2) \cdot 20 \cdot 20 - (1/2) \cdot 10 \cdot 10 = 150$, giving $K = 1/150$.



- (b) $f_R(x)$, the marginal pdf of R is the cross-sectional area of the joint pdf at x . Now, the cross-section is always a rectangle of height $1/150$, and its base is 10 if $0 < x < 10$, while the base is $20-x$ if $10 \leq x < 20$. Hence, we get that $f_R(x) = \begin{cases} 1/150, & 0 < x < 10, \\ (20-x)/150, & 10 \leq x < 20, \\ 0, & \text{elsewhere.} \end{cases}$ which is shown in the middle figure above. Note that

- $P\{\text{student spends less than 10 hours studying for the ECE 413 Final}\} = 2/3$.
- (c) T takes on values from 0 (when $S = 0$ and $R = 20$, that is, the student is sleepless and hopelessly confused) to 100 (when $S = 20$ and $R = 0$, that is, the student is well-rested and relies entirely on work done during the semester to carry him/her through the Final Exam). For any u , $0 < u < 100$, $P\{T \leq u\} = P\{S - R \leq (u-50)/2.5\} = (1/150) \cdot \text{shaded trapezoidal area in third figure above}$, but note that there are special cases to be considered when $u < 25$ and $u > 75$. We can now proceed to find $P\{T \leq u\} = F_T(u)$ and differentiate to find the pdf, but an easier argument is as follows.

For $25 \leq u < 75$, and small δu , $P\{u \leq T \leq u + \delta u\} \approx f_T(u) \cdot \delta u$

= volume between the two lines close together in the diagram = $(1/150) \cdot (10/\sqrt{2}) \cdot (\delta u/\sqrt{2})$, since that thin sliver has height $(1/150)$, and the sides of the base are $10/\sqrt{2}$ and $(\delta u/2.5)/\sqrt{2}$ respectively. We conclude that $f_T(u) = 1/75$ for $25 < u < 75$. A similar argument shows that the pdf of T increases linearly from 0 at $u = 0$ to $1/75$ at $u = 25$, and decreases from $1/75$ at $u = 75$ to 0 at $u = 100$. In short, it looks as shown above in the rightmost figure.

82.(a) $E[Z] = 2\{E[X^2] - E[Y^2]\} = 2\{\text{var}(X) - \text{var}(Y) + E^2[X] - E^2[Y]\} = -40$.

(b) $\text{cov}(T, U) = \text{cov}(2X+Y, 2X-Y) = 4 \cdot \text{var}(X) - \text{var}(Y) = 7$.

(c) $E[W] = E[3X+Y+2] = 3E[X] + E[Y] + 2 = 9$.

$\text{var}(W) = \text{var}(3X+Y+2) = \text{var}(3X+Y)$ (why?) = $9 \cdot \text{var}(X) + \text{var}(Y) + 6 \cdot \text{cov}(X, Y) = 9 \cdot 4 + 9 + 6 \cdot (0.1 \cdot 2 \cdot 3) = 48.6$.

(d) $P\{W > 0\} = 1 - \Phi((0-9)/\sqrt{48.6}) = 1 - \Phi(-9/\sqrt{48.6}) = \Phi(9/\sqrt{48.6})$ (why?)

- 83.(a) $\text{var}(\mathbf{X}+\mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) + 2 \cdot \text{cov}(\mathbf{X}, \mathbf{Y}) = 36$
 $\text{var}(\mathbf{X}-\mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) - 2 \cdot \text{cov}(\mathbf{X}, \mathbf{Y}) = 64$. Hence, $\text{cov}(\mathbf{X}, \mathbf{Y}) = -7$.
If $\text{var}(\mathbf{X}) = 3 \cdot \text{var}(\mathbf{Y})$, then $\text{var}(\mathbf{X}+\mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) + 2 \cdot \text{cov}(\mathbf{X}, \mathbf{Y}) = 4 \cdot \text{var}(\mathbf{Y}) - 14 = 36$, so that
 $\text{var}(\mathbf{Y}) = 12.5$, $\text{var}(\mathbf{X}) = 37.5$, and $\rho = \text{cov}(\mathbf{X}, \mathbf{Y}) / \sqrt{\text{var}(\mathbf{X})\text{var}(\mathbf{Y})} = -7 / (12.5\sqrt{3})$.
- (b) If $\text{var}(\mathbf{X}+\mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) + 2 \cdot \text{cov}(\mathbf{X}, \mathbf{Y}) = \text{var}(\mathbf{X}-\mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) - 2 \cdot \text{cov}(\mathbf{X}, \mathbf{Y})$, then it must be that $\text{cov}(\mathbf{X}, \mathbf{Y}) = 0$ and hence $\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated.
- (c) The values of $\text{var}(\mathbf{X})$ and $\text{var}(\mathbf{Y})$ do not affect the covariance computation at all. So, the random variables may or may not be uncorrelated.