

71.

$$\begin{aligned}
P(X_1 > X_2 + X_3) &+ P(X_2 > X_1 + X_3) + P(X_3 > X_1 + X_2) \\
&= 3P(X_1 > X_2 + X_3) \\
&= 3 \int_0^1 \int_0^{1-x_3} \int_{x_2+x_3}^1 dx_1 dx_2 dx_3 = 3 \int_0^1 \int_0^{1-x_3} (1 - x_2 - x_3) dx_2 dx_3 \\
&= 3 \int_0^1 \frac{(1-x_3)^2}{2} dx_3 = \frac{1}{2}
\end{aligned}$$

72.

$$\begin{aligned}
P(\min(X_1, \dots, X_n) > t) &= P(X_1 > t, \dots, X_n > t) = P(X_1 > t) \cdots P(X_n > t) \\
&= e^{-\lambda t} \cdots e^{-\lambda t} = e^{-n\lambda t}
\end{aligned}$$

showing that the minimum is exponential with rate $n\lambda$.

73. If we let X_i denote the time between the i^{th} and $(i+1)^{\text{st}}$ failure, $i = 0, 1, \dots, n-2$, then it follows from problem 72 that the X_i are independent exponential random variables with rate 2λ . Hence, $\sum_{i=0}^{n-2} X_i$ the amount of time the light can operate is gamma distributed with parameters $(n-1, 2\lambda)$.

74. (a)

$$P(X_1 > X_2 | X_1 > X_3) = \frac{P(X_1 = \max X_1, X_2, X_3)}{P(X_1 > X_3)} = \frac{1/3}{1/2} = 2/3$$

(b)

$$P(X_1 > X_2 | X_1 < X_3) = \frac{P(X_3 > X_1 > X_2)}{P(X_1 < X_3)} = \frac{1/3!}{1/2} = 1/3$$

(c)

$$P(X_1 > X_2 | X_2 > X_3) = \frac{P(X_1 > X_2 > X_3)}{P(X_2 > X_3)} = \frac{1/3!}{1/2} = 1/3$$

(d)

$$P(X_1 > X_2 | X_2 < X_3) = \frac{P(X_2 = \min X_1, X_2, X_3)}{P(X_2 < X_3)} = \frac{1/3}{1/2} = 2/3$$

75. (a)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = C f_{X,Y}(x,y)$$

where C does not depend on x . Writing

$$\left(\frac{x - \mu_x}{\sigma_x}\right)^2 - \frac{2\rho(x - \mu_x)(y - \mu_y)}{\sigma_x \sigma_y} = \frac{1}{\sigma_x^2} \left[x - \left(\mu_x + \rho(y - \mu_y) \frac{\sigma_x}{\sigma_y} \right) \right]^2 + K$$

where K does not depend on x , yields that given $Y = y$, X is normal with mean $\mu_x + \rho(y - \mu_y) \frac{\sigma_x}{\sigma_y}$ and variance $\sigma_x^2(1 - \rho^2)$.

- (b) That Y is normal with mean μ_y and variance σ_y^2 follows from (a) since

$$f_Y(y) = \frac{f_{X,Y}(x,y)}{f_{X|Y}(x|y)}.$$

The result for X can be proven in a similar fashion.

- (c) Follows since the joint density factors when $\rho = 0$.

76. Let $Y = \max X_1, \dots, X_n$ and $Z = \min X_1, \dots, X_n$. Then

$$P(Y \leq x) = P(X_1 \leq x, \dots, X_n \leq x) = \prod_{i=1}^n P(X_i \leq x) = F^n(x)$$

and

$$P(Z > x) = P(X_1 > x, \dots, X_n > x) = \prod_{i=1}^n P(X_i > x) = [1 - F(x)]^n$$

77. Let

$$I_j = \begin{cases} 1, & \text{success on trial } j \\ 0, & \text{otherwise} \end{cases}$$

Then

$$E \left[\sum_{j=1}^n I_j \right] = \sum_{j=1}^n E[I_j] = \sum_{j=1}^n p_j \quad (\text{independence not needed})$$

and

$$\text{var} \left(\sum_{j=1}^n I_j \right) = \sum_{j=1}^n \text{var}(I_j) = \sum_{j=1}^n p_j(1 - p_j) \quad (\text{independence needed})$$