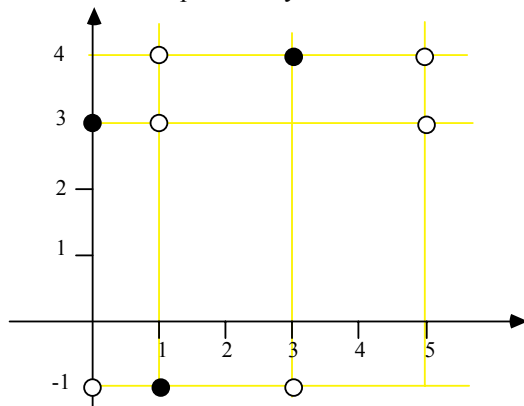


67.(a) In the figure below, I use $\circ$ to denote a probability mass of $1/12$ and $\bullet$ to denote a probability mass of $1/6$.



- (a) $p_X(0) = 1/6 + 1/12 = 1/4$, $p_X(1) = 1/3$, $p_X(3) = 1/4$, $p_X(5) = 1/6$.
 $p_Y(-1) = 2 \times 1/12 + 1/6 = 1/3$, $p_Y(3) = 1/3$, $p_Y(4) = 1/3$.
- (b) Since, for example, $p_X(3)p_Y(3) = (1/3) \times (1/4) = 1/12 \neq p_{X,Y}(3,3) = 0$, the random variables are not independent. We can also see this via the eyeball test: there are empty spots in the grid.
- (c) $P\{X \leq Y\} = p_{X,Y}(0,3) + p_{X,Y}(1,3) + p_{X,Y}(1,4) + p_{X,Y}(3,4) = 1/2$.
 $P\{X + Y \leq 8\} = 1 - p_{X,Y}(5,4) = 11/12$.
- (d) Given $Y = 3$, X takes on values 0, 1 and 5 with conditional probabilities $(1/6)/(1/3)$, $(1/12)/(1/3)$, and $(1/12)/(1/3)$, that is, $1/2$, $1/4$ and $1/4$ respectively. Hence, $E[X|Y=3] = 0 \cdot (1/2) + 1 \cdot (1/4) + 5 \cdot (1/4) = 3/2$, and $E[X^2|Y=3] = 0^2 \cdot (1/2) + 1^2 \cdot (1/4) + 5^2 \cdot (1/4) = 26/4$, giving $\text{var}(X|Y=3) = 26/4 - 9/4 = 17/4$.
- 68.(a) $p_X(i) = \sum p_{X,Y}(i,j) = 2^{-(i-1)} \cdot (1/3) \cdot [1 + 1/3 + (1/3)^2 + \dots] = 2^{-i}$, $i = 1, 2, \dots$. This is a geometric pmf with parameter $1/2$.
 $p_Y(j) = \sum p_{X,Y}(i,j) = 3^{-j} \times [1 + (1/2) + (1/2)^2 + \dots] = 3^{-j} \times 2 = (2/3)(1/3)^{j-1}$, $j = 1, 2, \dots$. This is a geometric pmf with parameter $2/3$.
- (b) $p_{X|Y}(i|k) = p_{X,Y}(i,k)/p_Y(k) = 2^{-(i-1)} 3^{-k} / 3^{-k} \times 2 = 2^{-i}$, $i = 1, 2, \dots$. This is a geometric pmf with parameter $1/2$.
- (c) Yes, X and Y are independent random variables since $p_{X,Y}(i,j) = 2^{-(i-1)} 3^{-j} = p_X(i)p_Y(j) = 2^{-i} \times 3^{-j} \times 2$. This can also be deduced from the fact that the conditional pmf of X is the same as the unconditional pmf.
69. See also Lecture 31. The pmf of Y is Poisson with parameter λp . If $Y = k$, then there must have been at least k α -particles emitted by the source, i.e. we know that X takes on values k and larger only. It turns out that, conditioned on $Y = k$, we can express X as $k + Z$ where Z is a Poisson random variable with parameter $\lambda(1-p)$. The conditional pmf of X given $Y = k$ is thus a Poisson pmf shifted k places to the right: $P\{X = k+n\} = \exp(-\lambda(1-p))[\lambda(1-p)]^n/n!$ for $n = 0, 1, 2, \dots$.
70. The joint pdf is nonzero on the quarter-plane shown. For $v > 0$, the pdf of Y is given by
- $$f_Y(v) = \int_{u=-v}^v c \cdot (v^2 - u^2) \cdot \exp(-v) du = c \cdot [v^2 u - u^3/3] \cdot \exp(-v) \Big|_{-v}^v = c \cdot (4/3) \cdot v^3 \cdot \exp(-v).$$
- We recognize this as a gamma pdf with parameters $(4,1)$ which gives us that $c \cdot (4/3) = 1/\Gamma(4) = 1/3!$ and thus $c = 1/8$.
- For $u > 0$, $f_X(u) = \int_{v=u}^{\infty} (1/8) \cdot (v^2 - u^2) \cdot \exp(-v) dv = (1/8) \cdot (-v^2 - 2v - 2 + u^2) \cdot \exp(-v) \Big|_u^{\infty} = (1/4) \cdot (1+u) \cdot \exp(-u)$ while for $u < 0$, the integral is from $v = -u$ to ∞ . Thus, we get $f_X(u) = (1/4) \cdot (1+|u|) \cdot \exp(-|u|)$.