

53.(a),(b) As studied in class, X is an exponential random variable with parameter μ . Hence, $P\{X > \tau\} = \exp(-\mu\tau)$ and $f_X(u) = \mu \cdot \exp(-\mu u)$ for $u > 0$, and 0 otherwise.

(c) The number of arrivals in the time interval $(0, 6]$ is a Poisson random variable $N(0,6]$ with parameter 6μ , and we get $P\{N(0,6] = 4\} = \frac{(6\mu)^4}{4!} \cdot \exp(-6\mu)$.

(d) The number of arrivals in $(4, 10]$ is $N(4,10]$ is also a Poisson random variable with parameter 6μ , and hence $P\{N(4,10] = 2\} = \frac{(6\mu)^2}{2!} \cdot \exp(-6\mu)$. We are now asked for $P[\{N(0,6] = 4\} \cap \{N(4,10] = 2\}]$. Now, $(0, 6]$ and $(4, 10]$ are intersecting intervals of time, but we can partition them into the disjoint intervals $(0, 4]$, $(4, 6]$, and $(6,10]$ and thus write

$$\begin{aligned} P[\{N(0,6] = 4\} \cap \{N(4,10] = 2\}] &= P[\{N(0,4] = 4\} \cap \{N(4,6] = 0\} \cap \{N(6,10] = 2\}] \\ &+ P[\{N(0,4] = 3\} \cap \{N(4,6] = 1\} \cap \{N(6,10] = 1\}] \\ &+ P[\{N(0,4] = 2\} \cap \{N(4,6] = 2\} \cap \{N(6,10] = 0\}] \\ &= \left[\frac{(4\mu)^4}{4!} \cdot \exp(-4\mu) \right] \cdot [\exp(-2\mu)] \cdot \left[\frac{(4\mu)^2}{2!} \cdot \exp(-4\mu) \right] \\ &+ \left[\frac{(4\mu)^3}{3!} \cdot \exp(-4\mu) \right] \cdot \left[\frac{2\mu}{1!} \exp(-2\mu) \right] \cdot \left[\frac{4\mu}{1!} \exp(-4\mu) \right] \\ &+ \left[\frac{(4\mu)^2}{2!} \cdot \exp(-4\mu) \right] \cdot \left[\frac{(2\mu)^2}{2!} \cdot \exp(-2\mu) \right] \cdot [\exp(-4\mu)] \end{aligned}$$

because occurrences in disjoint intervals are independent and hence we can multiply the probabilities.

$$\text{Thus, } P[\{N(0,6] = 4\} \cap \{N(4,10] = 2\}] = \left(\frac{256}{3}\right) \cdot \exp(-10\mu) \cdot \left[\mu^6 + \mu^5 + \left(\frac{3}{16}\right)\mu^5\right].$$

(e) $P\{X > \tau \mid A\}$ is obviously 0 if $\tau \geq 6$, while for $0 \leq \tau < 6$,
 $P\{X > \tau \mid A\} = P\{\text{No arrivals in } (0, \tau] \mid 4 \text{ arrivals in } (0, 6]\}$
 $= \frac{[\exp(-\mu\tau) \cdot \exp(-\mu(6-\tau)) \cdot (\mu(6-\tau))^4 / 4!]}{\exp(-6\mu) \cdot (6\mu)^4 / 4!} = \frac{(6-\tau)^4}{6^4} = \left(\frac{6-\tau}{6}\right)^4$ which has value 1 at $\tau = 0$ and approaches 0 as τ approaches 6.

54.(a) (a) $\Phi(.8333) = .7977$, (b) $2\Phi(1)-1 = .6827$, (c) $1-\Phi(.3333) = .3695$, (d) $\Phi(1.6667) = .9522$, (e) $1-\Phi(1) = .1587$

(b) The final step of parts (a) and (b) use that $-Z$ is also a standard normal random variable.

$$(a) P\{Z > x\} = P\{-Z < -x\} = P\{Z < -x\}$$

$$(b) P\{|Z| > x\} = P\{Z > x\} + P\{Z < -x\} = P\{Z > x\} + P\{-Z > x\} = 2P\{Z > x\}$$

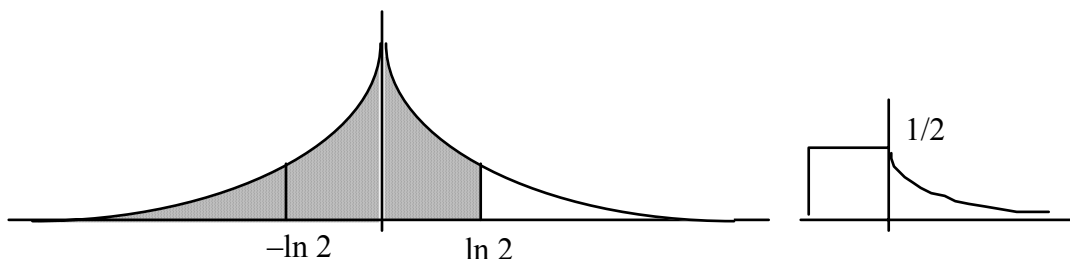
$$(c) P\{|Z| < x\} = 1 - P\{|Z| > x\} = 1 - 2P\{Z > x\} = 1 - 2(1 - P\{Z < x\}) = 2P\{Z < x\} - 1$$

55. $P\{X > 50\} = P\{(X-40)/4 > 10/4\} = 1 - \Phi(2.5) = 1 - .9938$. Hence, $(P\{X < 50\})^{10} = (.9938)^{10}$.

56.

$$P\{\text{inblack} \mid 5\} = \frac{P\{5 \mid \text{black}\}\alpha}{P\{5 \mid \text{black}\}\alpha + P\{5 \mid \text{white}\}(1-\alpha)} = \frac{\frac{1}{2\sqrt{2\pi}} e^{-(5-4)^2/8} \alpha}{\frac{1}{2\sqrt{2\pi}} e^{-(5-4)^2/8} \alpha + (1-\alpha) \frac{1}{3\sqrt{2\pi}} e^{-(5-6)^2/18}} = \frac{\frac{\alpha}{2} e^{-1/8}}{\frac{\alpha}{2} e^{-1/8} + \frac{1-\alpha}{3} e^{-1/18}}$$

α is the value that makes preceding equal 1/2.



57.(a) From the figure shown above, we see that the pdf is symmetric about $u = 0$. Hence, we get that

$$P\{X \leq \ln 2\} = \frac{1}{2} + \int_0^{\ln 2} \frac{1}{2} \exp(-u) du = \frac{1}{2} + \left[-\frac{1}{2} \exp(-u) \right]_0^{\ln 2} = \frac{3}{4}. \text{ Notice that } P\{0 \leq X \leq \ln 2\} = \frac{1}{4}.$$

$$(b) P\{|X| \leq \ln 2 \mid X \leq \ln 2\} = \frac{P\{|X| \leq \ln 2\} \cap \{X \leq \ln 2\}}{P\{X \leq \ln 2\}} = \frac{P\{X \leq \ln 2\}}{P\{X \leq \ln 2\}} = \frac{2P\{0 \leq X \leq \ln 2\}}{3/4} = \frac{1/2}{3/4} = \frac{2}{3}.$$

$$(c) \cos(\pi X/2) = 0 \text{ if } X \text{ is an odd integer, and} \\ P\{\cos(\pi X/2) < 0\} = \dots + P\{-7 < X < -5\} + P\{-3 < X < -1\} + P\{1 < X < 3\} + P\{5 < X < 7\} + \dots \\ = 2[P\{1 < X < 3\} + P\{5 < X < 7\} + \dots] \text{ (by symmetry)} = [\exp(-1) - \exp(-3)] + [\exp(-5) - \exp(-7)] + \dots \\ = \exp(-1) \cdot \frac{1}{1 + \exp(-2)} = \frac{1}{2 \cdot \cosh(1)} = 0.324 \dots \text{ on summing the geometric series.}$$

$$(d) \text{ The minimum value of } \mathbf{I} \text{ is } -1. F_{\mathbf{I}}(b) = P\{\mathbf{I} \leq b\} = 0 \text{ if } b < -1. \\ \text{For } b \geq -1, F_{\mathbf{I}}(b) = P\{\mathbf{I} \leq b\} = P\{e^{\mathbf{V}} - 1 \leq b\} = P\{\mathbf{V} \leq \ln(b+1)\} = F_{\mathbf{V}}(\ln(b+1)). \text{ But,}$$

$$F_{\mathbf{V}}(a) = \begin{cases} e^{a/2}, a \leq 0, \\ 1 - e^{-a/2}, a > 0. \end{cases} \text{ Thus, } F_{\mathbf{I}}(b) = \begin{cases} \frac{b+1}{2} - 1 \leq b \leq 0, \\ 1 - \frac{1}{2(b+1)}, b > 0, \end{cases}$$

$$\text{and } f_{\mathbf{I}}(b) = \begin{cases} \frac{1}{2} - 1 \leq b \leq 0, \\ \frac{1}{2(b+1)^2}, b > 0. \end{cases} \text{ Note that the pdf has constant value } 1/2 \text{ in the range } -1 \leq b \leq 0.$$

$$58.(a),(b) \text{ Let } 0 \leq v \leq 1. \text{ Then, } F_{\mathbf{Y}}(v) = P\{\mathbf{Y} \leq v\} = P\{(1 - \mathbf{X})^2 \leq v\} = P\{-\sqrt{v} \leq 1 - \mathbf{X} \leq \sqrt{v}\} = P\{\mathbf{X} \geq 1 - \sqrt{v}\} \\ = 1 - F_{\mathbf{X}}(1 - \sqrt{v}) = (1 - (1 - \sqrt{v}))^2 = v \text{ where we used the result that } F_{\mathbf{X}}(u) = 1 - (1 - u)^2 \text{ for } 0 \leq u \leq 1.$$

$$\text{Hence, } F_{\mathbf{Y}}(v) = \begin{cases} 0, v < 0, \\ v, 0 \leq v \leq 1, \\ 1, v > 1. \end{cases} \text{ and } f_{\mathbf{Y}}(v) = \begin{cases} 1, 0 \leq v \leq 1 \\ 0, \text{elsewhere.} \end{cases}.$$