

45. $E[\text{Points}] = 10(1/10) + 5(2/10) + 3(2/10) = 2.6$

46. (a)

$$\begin{aligned} E[|X - a|] &= \int_a^A \frac{x - a}{A} dx + \int_0^a \frac{a - x}{A} dx = \frac{A}{2} - \left(a - \frac{a^2}{A}\right) \\ \frac{d}{da}(\cdot) &= \frac{2a}{A} - 1 = 0 \Rightarrow a = A/2 \end{aligned}$$

(b)

$$\begin{aligned} E[|X - a|] &= \int_0^\infty (a - x)\lambda e^{-\lambda x} dx + \int_a^A (x - a)\lambda e^{-\lambda x} dx \\ &= a(1 - e^{-\lambda a}) + ae^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda} - \frac{1}{\lambda} + ae^{-\lambda a} + \frac{e^{-\lambda a}}{\lambda} - ae^{-\lambda a} \end{aligned}$$

Differentiation yields that the minimum is attained at $\bar{a}$ where

$$e^{-\lambda \bar{a}} = 1/2 \text{ or } \bar{a} = \log(2/\lambda)$$

47. Since the exponential distribution is memoryless, the probability that the radio will be working after an additional years is $1 - F(8) = e^{-1}$.

48.

$$\begin{aligned} E[X^k] &= \int_0^\infty x^k \lambda e^{-\lambda x} dx \\ &= \lambda^{-k} \int_0^\infty (\lambda x)^k \lambda e^{-\lambda x} dx \\ &= \lambda^{-k} \Gamma(k + 1) = k!/\lambda^k \end{aligned}$$

49. Using Equation (6.3) on p. 222 of Ross,

$$\begin{aligned} E[X] &= \frac{B(a+1, b)}{B(a, b)} = \frac{\Gamma(a+1)}{\Gamma(a+b+1)} \frac{\Gamma(a+b)}{\Gamma(a)} = \frac{a}{a+b} \\ E[X^2] &= \frac{B(a+2, b)}{B(a, b)} = \frac{\Gamma(a+2)}{\Gamma(a+b+2)} \frac{\Gamma(a+b)}{\Gamma(a)} = \frac{(a+1)a}{(a+b+1)(a+b)} \end{aligned}$$

Thus,

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{(a+1)a}{(a+b+1)(a+b)} - \frac{a^2}{(a+b)^2} = \frac{ab}{(a+b+1)(a+b)^2}$$

50.

$$\begin{aligned} E[X] &= \int_0^{\infty} x 2(e^{-3x} + e^{-6x}) dx = 2 \left(\int_0^{\infty} x e^{-3x} dx + \int_0^{\infty} x e^{-6x} dx \right) \\ &= 2 \left(\frac{1}{3 \cdot 3} + \frac{1}{6 \cdot 6} \right) = \frac{5}{18} \end{aligned}$$

Likewise,

$$\begin{aligned} E[X^2] &= \int_0^{\infty} x^2 2(e^{-3x} + e^{-6x}) dx = 2 \left(\int_0^{\infty} x^2 e^{-3x} dx + \int_0^{\infty} x^2 e^{-6x} dx \right) \\ &= 2 \left(\frac{2}{3 \cdot 9} + \frac{2}{6 \cdot 36} \right) = \frac{7}{54} \end{aligned}$$

Hence,

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{7}{54} - \frac{5^2}{18^2} = \frac{17}{324}$$

Note that $P(|X| < 2) = P(X < 2)$ since X is positive valued. Therefore,

$$\begin{aligned} P(|X| < 2) &= \int_0^2 2(e^{-3x} + e^{-6x}) dx = 2 \left(\int_0^2 e^{-3x} dx + \int_0^2 e^{-6x} dx \right) \\ &= 2 \left(\frac{1}{3} e^{-2/3} + \frac{1}{6} e^{-2/6} \right) \end{aligned}$$

51.

$$\begin{aligned} E[X] &= \int_{-\infty}^1 x f(x) dx + 1 \cdot P(X = 1) + \int_1^2 x f(x) dx + 2 \cdot P(X = 2) + \int_2^{\infty} x f(x) dx \\ &= 1 \cdot \frac{1}{3} + \int_1^2 \frac{x}{3} dx + 2 \cdot \frac{1}{3} = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} E[X^2] &= \int_{-\infty}^1 x^2 f(x) dx + 1 \cdot P(X = 1) + \int_1^2 x^2 f(x) dx + 4 \cdot P(X = 2) + \int_2^{\infty} x^2 f(x) dx \\ &= 1 \cdot \frac{1}{3} + \int_1^2 \frac{x^2}{3} dx + 4 \cdot \frac{1}{3} = \frac{22}{9} \end{aligned}$$

Hence,

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{7}{36}.$$

Note that $P(|X - 1| < 1) = P(0 < X < 2)$. Therefore,

$$P(|X - 1| < 1) = \int_0^1 f(x) dx + P(X = 1) + \int_1^2 f(x) dx = \int_0^1 \frac{1}{3} dx + \frac{1}{3} = \frac{2}{3}$$

52.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^0 ce^{2x}dx + \int_0^{\infty} ce^{-2x}dx = 2 \int_0^{\infty} ce^{-2x}dx = c = 1$$

Observe that $E[X] = 0$ since the density function is symmetric, i.e. $f(x) = f(-x)$. By linearity of expectation $E[Y] = E[3X - 2] = 3E[X] - 2 = -2$.

$$E[X^2] = \int_{-\infty}^0 x^2 e^{2x} dx + \int_0^{\infty} x^2 e^{-2x} dx = 2 \int_0^{\infty} x^2 e^{-2x} dx = \frac{1}{4}$$

Since $E[X] = 0$ we have $Var(X) = E[X^2] = \frac{1}{4}$. Then $Var(Y) = Var(3X - 2) = 9Var(X) = \frac{9}{4}$.

Note that $P(Y > 4) = P(X > 2)$ and $P(Y > 0) = P(X > \frac{2}{3})$.

$$\begin{aligned} P(X > 2) &= \int_2^{\infty} e^{-2x} dx = \frac{1}{2}e^{-4} \\ P(X > \frac{2}{3}) &= \int_{\frac{2}{3}}^{\infty} e^{-2x} dx = \frac{1}{2}e^{-4/3} \end{aligned}$$

Therefore,

$$P(Y > 4 | Y > 0) = \frac{e^{-4}}{e^{-4/3}} = e^{-8/3}$$