

31. (a) All we know when the procedure ends is that the two last flips were either  $HT$  or  $TH$ . Thus,

$$\begin{aligned} P(\text{heads}) &= P(HT|HT \text{ or } TH) \\ &= \frac{P(HT)}{P(HT) + P(TH)} = \frac{p(1-p)}{p(1-p) + (1-p)p} = \frac{1}{2} \end{aligned}$$

- (b) No, with this new procedure the result will be heads (tails) whenever the first flip is tails (heads). Hence, it will be heads with probability  $1-p$ .
32. Conditioning on the outcome of the initial trial

$$\begin{aligned} P(E \text{ before } F) &= P(E \text{ before } F|E)P(E) + P(E \text{ before } F|F)P(F) \\ &\quad + P(E \text{ before } F| \text{ neither } E \text{ or } F)[1 - P(E) - P(F)] \end{aligned}$$

We have  $P(E \text{ before } F|E) = 1$ ,  $P(E \text{ before } F|F) = 0$ , and  $P(E \text{ before } F| \text{ neither } E \text{ or } F) = P(E \text{ before } F)$  by independence. Hence,

$$P(E \text{ before } F) = \frac{E}{P(E) + P(F)}$$

33. Using DeMorgan's Law and the fact that if  $(E_i)_{i=1}^n$  are independent events so are  $(E_i^C)_{i=1}^n$ , we get

$$P(\cup_{i=1}^n E_i) = 1 - P(\cap_{i=1}^n E_i^C) = 1 - \prod_{i=1}^n [1 - P(E_i)].$$

34.  $P(A) = P(B) = P(C) = 1/2$ ,  $P(AB) = P(AC) = P(BC) = 1/4$ .  
But  $P(ABC) = 1/4$ .
35. (a) We do not have sufficient information to conclude one way or another.  
(b) Since  $P(ABC) \neq P(A)P(B)P(C)$ ,  $A, B, C$  are not independent.  
(c) Since  $P(BC) \neq P(B)P(C)$ ,  $A, B, C$  are not independent.
36. Let the two hypotheses be  
 $H_0$ : Batch manufactured in production line A ( $P(\text{defective}|H_0) = p_0 = 0.05$ )  
 $H_1$ : Batch manufactured in production line B ( $P(\text{defective}|H_1) = p_1 = 0.10$ )

Let  $X$  be the number of defective items in a batch of size  $N$ . For  $i = 0, 1$  if hypothesis  $H_i$  is true, then  $X$  is a binomial random variable with parameters  $N$  and  $p_i$ . If  $X = D$  defectives are observed the likelihood ratio is given by

$$\Lambda(D) = \frac{P(X = D|H_1)}{P(X = D|H_0)} = \frac{\binom{N}{D} p_1^D (1-p_1)^{N-D}}{\binom{N}{D} p_0^D (1-p_0)^{N-D}} = \frac{p_1^D (1-p_1)^{N-D}}{p_0^D (1-p_0)^{N-D}}$$

and the ML decision rule can be written as

$$\Lambda(D) \begin{cases} > 1 & \text{decide } H_1 \text{ is true} \\ < 1 & \text{decide } H_0 \text{ is true} \end{cases}$$

Equivalently, taking logarithms (note that the logarithm is a strictly monotone increasing function) we can express the decision rule in terms of the log-likelihood ratio

$$\lambda(D) \begin{cases} > 0 & \text{decide } H_1 \text{ is true} \\ < 0 & \text{decide } H_0 \text{ is true} \end{cases}$$

where

$$\begin{aligned} \lambda(D) &= \log(\Lambda(D)) = D \log\left(\frac{p_1}{p_0}\right) + (N - D) \log\left(\frac{1 - p_1}{1 - p_0}\right) \\ &= D(1 - \log(\frac{18}{19})) + N \log(\frac{18}{19}) \end{aligned}$$

Finally, the decision rule in terms of  $D$  is

$$D \begin{cases} > 0.0724N & \text{decide } H_1 \text{ is true} \\ < 0.0724N & \text{decide } H_0 \text{ is true} \end{cases}$$

For  $N = 100$ ,  $D = 7$  the decision should be  $H_0$ , it is more likely that the batch has been manufactured in production line A.

37. Let the two hypotheses be  
 $H_0$ : coin is fair,  $P(\text{Head}|H_0) = p_0 = 0.5$   
 $H_1$ : coin is biased,  $P(\text{Head}|H_1) = p_1 = 0.7$

Let  $X$  be the number of heads observed.  $X$  is a binomial random variable with parameters 5 and  $p_0$  or  $p_1$  depending on the hypothesis. In particular, we have

$$P(X = k|H_i) = \binom{5}{k} p_i^k (1 - p_i)^{5-k}, \quad \text{for } k = 0, 1, 2, 3, 4, 5 \text{ and } i = 0, 1.$$

- (a) The likelihood matrix is given below

|                | X=0           | X=1           | X=2           | X=3           | X=4           | X=5           |
|----------------|---------------|---------------|---------------|---------------|---------------|---------------|
| $P(X = k H_0)$ | <u>0.0313</u> | <u>0.1563</u> | <u>0.3125</u> | <u>0.3125</u> | 0.1563        | 0.0313        |
| $P(X = k H_1)$ | 0.0024        | 0.0284        | 0.1323        | 0.3087        | <u>0.3601</u> | <u>0.1681</u> |

The ML rule is to decide that the coin is biased when 4 or more heads appear. The probability of making a wrong decision is 0.2444.

- (b) The joint probability matrix is given below

|                 | X=0           | X=1           | X=2           | X=3           | X=4           | X=5           |
|-----------------|---------------|---------------|---------------|---------------|---------------|---------------|
| $P(X = k, H_0)$ | <u>0.0250</u> | <u>0.1250</u> | <u>0.2500</u> | <u>0.2500</u> | <u>0.1250</u> | 0.0250        |
| $P(X = k, H_1)$ | 0.0005        | 0.0057        | 0.0265        | 0.0617        | 0.0720        | <u>0.0336</u> |

The MAP rule is to decide that the coin is biased when 5 heads appear. The probability of making a wrong decision is 0.1914.

- (c) One should use the MAP rule whenever the prior probabilities, i.e.  $\pi_0 = P(H_0)$  and  $\pi_1 = P(H_1)$ , are available since the MAP rule minimizes the probability of decision error.