

23. If strategy (a) is used, they will win with probability p . If strategy (b) is used, then

$$\begin{aligned} P\{\text{win}\} &= P\{\text{win}|\text{both correct}\}p^2 + P\{\text{win}|\text{exactly one correct}\}2p(1-p) \\ &\quad + P\{\text{win}|\text{neither correct}\}(1-p)^2 \\ &= p^2 + p(1-p) + 0 = p \end{aligned}$$

Thus, both strategies give the same probability of winning.

24. Let Y denote the largest of the remaining m chips. By exercise 28

$$P\{Y = j\} = \frac{\binom{j-1}{m-1}}{\binom{m+n}{m}}, \quad m \leq j \leq n+m$$

Now, $X = n + m - Y$ and so

$$P\{X = i\} = P\{Y = m + n - i\} = \frac{\binom{m+n-i-1}{m-1}}{\binom{m+n}{m}}, \quad i \leq n$$

25. (a) $P(A|B^c) = 1 - P(A^c|B^c) = 0.6$. $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = 0.39$.
 $P(B|A) = P(AB)/P(A) = P(A|B)P(B)/P(A) = 0.21/0.39 = 7/13$.
 (b) $1 \geq P(G \cup H) = P(G) + P(H) - P(GH)$. Hence, $P(GH) \geq 2/3 + 2/3 - 1 = 1/3$,
 and $P(G|H) = P(GH)/P(H) = P(GH)/(2/3) \geq (1/3)/(2/3) = 1/2$.

26. Given that $X = n$, we know that there was a success on the n -th trial, and $r-1$ successes in the previous $n-1$ trials so that $P\{X = n\} = p \binom{n-1}{r-1} p^{r-1} (1-p)^{(n-1)-(r-1)} = \binom{n-1}{r-1} p^r (1-p)^{n-r}$. If there is a success on the i -th trial also, then there are $r-2$ successes on the *other* $n-2$ trials preceeding the n -th one, i.e. $P(\{X = n\} \cap \{\text{success on } i\text{-th}\}) = pp \binom{n-2}{r-2} p^{r-2} (1-p)^{(n-2)-(r-2)} = \binom{n-2}{r-2} p^r (1-p)^{n-r}$. Hence, the conditional probability is the ratio of the binomial coefficients which works out to $(r-1)/(n-1)$. Alternatively, consider that given $X = n$ we know that there are $r-1$ successes in the previous $n-1$ trials. But, we saw in class that given that there are k successes on n trials, the probability of a success on the i -th trial is just k/n independent of p . So, here the answer must be $(r-1)/(n-1)$.

27. (a)

$$\begin{aligned} P\{X \leq 100\} &= 1 - P\{X > 100\} \\ &= 1 - [P\{X = 105\} + P\{X = 104\} + P\{X = 103\} \\ &\quad + P\{X = 102\} + P\{X = 101\}] \\ &= 0.9833 \end{aligned}$$

- (b) If X is a binomial random variable with parameters (n, p) , then $Y = n - X$ is a binomial random variable with parameters $(n, 1-p)$. Thus, the number of no-shows is a binomial random variable with parameters $(105, 0.1)$. Since n is large and p is small, it is reasonable to approximate this as a Poisson random variable with parameter $\lambda = np = 10.5$.

(c)

$$\begin{aligned}P\{Y \geq 5\} &= 1 - P\{Y < 5\} \\&= 1 - [P\{Y = 0\} + P\{Y = 1\} + P\{Y = 2\} + P\{Y = 3\} + P\{Y = 4\}] \\&= 1 - \exp^{-10.5}[1 + (10.5) + (10.5)^2/2 \\&\quad (10.5)^3/6 + (10.5)^4/24] \\&= 0.9789\end{aligned}$$

which is fairly close to the probability obtained in part (a).

28. Let A and B denote respectively the events that your first and your final choice is the curtain concealing the prize. $P(A) = 1/3, P(A^c) = 2/3$. Now, if you switch, $P(B|A) = 0$, while $P(B|A^c) = 1$. Hence, $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 2/3$. If you don't switch, on the other hand, then $P(B|A) = 1$, while $P(B|A^c) = 0$. Hence, $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 1/3$. So, given the choice you should switch.

29. (a)

$$1 - \sum_{n=1}^{\infty} \alpha p^n = 1 - \frac{\alpha p}{1 - p}$$

- (b) Condition on the number of children: For $k > 0$

$$\begin{aligned}P\{k \text{ boys}\} &= \sum_{n=1}^{\infty} P\{k|n \text{ children}\} \alpha p^n \\&= \sum_{n=k}^{\infty} \binom{n}{k} (1/2)^n \alpha p^n \\P\{0 \text{ boys}\} &= 1 - \frac{\alpha p}{1 - p} + \sum_{n=1}^{\infty} \alpha p^n (1/2)^n\end{aligned}$$

30. (a) The bill passes if the Conservative Republicans and at least one other group votes for it, i.e. the event $A \cap (B \cup C \cup D)$ occurs, or if all groups except the Conservative Republicans vote for it, i.e. the event $A^c \cap B \cap C \cap D$ occurs. Note that these are disjoint events. Now, $(B \cup C \cup D)^c = B^c \cap C^c \cap D^c$ by DeMorgan's theorem, and thus we can partition A into two disjoint events as follows:

$$A = A \cap (B \cup C \cup D) \cup A \cap B^c \cap C^c \cap D^c.$$

Hence, we see that

$$\begin{aligned}P(A \cap (B \cup C \cup D)) &= P(A) - P(A \cap B^c \cap C^c \cap D^c) \\&= P(A) - P(A)P(B^c)P(C^c)P(D^c)\end{aligned}$$

by independence. Thus,

$$\begin{aligned}P(\text{bill passes}) &= P(A \cap (B \cup C \cup D)) + P(A^c \cap B \cap C \cap D) \\&= P(A) - P(A)P(B^c)P(C^c)P(D^c) + P(A^c)P(B)P(C)P(D) \\&= 0.9 \times [1 - 0.4 \times 0.5 \times 0.8] + 0.1 \times 0.6 \times 0.5 \times 0.2 \\&= 0.756 + 0.006 = 0.762\end{aligned}$$

- (b) The Conservative Republicans must be in favor of the motion to override to pass. Furthermore, either the Liberal Democrats must also be in favor or both the other groups must also be in favor. Hence, $P(\text{motion to override passes}) =$

$P(E(H \cup FG)) = P(E) \times P(H \cup FG)$ by independence of E and $(H \cup FG)$. We have (using independence)

$$\begin{aligned} P(\text{motion to override passes}) &= P(E) \times P(H \cup FG) \\ &= P(E) \times [P(H) + P(FG) - P(FGH)] \\ &= P(E) \times [P(H) + P(F)P(G) - P(F)P(G)P(H)] \\ &= 0.99 \times [0.1 + 0.4 \times 0.6 + 0.6 \times 0.4 \times 0.1] \\ &= 0.3128 \end{aligned}$$