

16. (a) and (b): (i) $\sum_{k=5}^9 \binom{9}{k} p^k (1-p)^{9-k}$,

(ii) $\sum_{k=5}^8 \binom{8}{k} p^k (1-p)^{8-k}$,

(iii) $\sum_{k=4}^7 \binom{7}{k} p^k (1-p)^{7-k}$

where $p = 0.7$ in (a) and $p = 0.3$ in (b).

(c) For $p = 0.6$, compare the probabilities in cases (i), (ii), (iii) and pick the minimum.

17. $P\{\text{wins in exactly } i \text{ games}\} = P\{\text{wins the last game, wins 3 out of } i-1 \text{ previous games}\} = 0.6 \times \binom{i-1}{3} (0.6)^3 (0.4)^{(i-1)-3} = \binom{i-1}{3} (0.6)^4 (0.4)^{i-4}$. Calculate this for $i = 4, 5, 6, 7$. $P\{\text{wins a 2-out-of-3 series}\} = P\{\text{WWX, WLW, LWW}\} = (0.6)^2 + 2 \times (0.6)^2 (0.4)$, where X stands for don't care, i.e. W (win) or L (lose).

18. $P\{\mathbf{X} \text{ is odd}\} = P\{\mathbf{X} = 1\} + P\{\mathbf{X} = 3\} + \dots + P\{\mathbf{X} = M\}$ where $M = N$ if N is odd $M = N - 1$ if N is even. Hence,

$$P\{\mathbf{X} \text{ is odd}\} = \binom{N}{1} q^{N-1} p + \binom{N}{3} q^{N-3} p^3 + \dots + \binom{N}{M} q^{N-M} p^M.$$

Note the last term is p^N if N is odd, and it is Nqp^{N-1} if N is even. Now,

$$(q+p)^N = q^N + \binom{N}{1} q^{N-1} p + \dots + \binom{N}{N-1} qp^{N-1} + p^N$$

$$(q-p)^N = q^N - \binom{N}{1} q^{N-1} p + \dots + (-1)^{N-1} \binom{N}{N-1} qp^{N-1} + (-1)^N p^N$$

We subtract the second equation from the first one and note that alternate terms cancel. Also if N is even, the last terms cancel, while if N is odd, the next-to-last-terms cancel. In either case, we see that the result is just $2P\{\mathbf{X} \text{ is odd}\}$, i.e.

$$P\{\mathbf{X} \text{ is odd}\} = [(q+p)^N - (q-p)^N]/2 = [1 - (1-2p)^N]/2.$$

19. $\mathbf{Y} = 1, 2, 3$ passengers are left behind according to $\mathbf{X} = 5, 6, 7$. Since $\mathbf{X}$ takes on values 5, 6, 7 with probabilities $\frac{21}{128}, \frac{7}{128}, \frac{1}{128}$ respectively, we readily find that $E[\mathbf{Y}] = \frac{1 \times 21 + 2 \times 7 + 3 \times 1}{128} = \frac{19}{64}$.

20. (a) Maximize your chances of living by predicting the most probable outcome. According to Proposition 6.1, p. 145 in Ross, the most probable number of Heads is largest integer less than or equal to $(N+1)p$ which is 110. So, predict that 110 Heads will occur. This has probability $\binom{1000}{110} (0.11)^{110} (0.89)^{890} = 0.04028$. Not great odds of survival! It is important to understand that the answer is *not* the mean value Np , even though both formulas happen to give the same number in this particular instance.

(b) $P\{\text{Head occurring for first time on the } k\text{-th toss}\} = (1-p)^{k-1} p$ is a *decreasing* function of k . So predict that Head occurs for the first time on the first toss, and survive with probability 0.11! Note that if you had chosen the mean value, i.e. $1/p = 9.09$ and rounded it off to 9, your chances of surviving would be only $(0.89)^8 0.11 = 0.0433$.

- (c) $P\{105\text{th head on } k\text{-th toss}\} = \binom{k-1}{104} p^{105} q^{k-105}$. To find which value of k maximizes this, consider the ratio

$$\frac{P\{105\text{th head on } k\text{-th toss}\}}{P\{105\text{th head on } (k-1)\text{-th toss}\}} = \frac{k-1}{k-105} q \geq 1$$

as long as $k \leq \frac{105q}{1-q} = 946.45$. So predict that 105th head will occur on the 946th toss. The probability of this event is $\binom{945}{104} p^{105} q^{841} = 0.00455$.

21. (a) $P\{\text{Head occurring for first time on 12th toss}\} = (1-p)^{11}p$. It is easy to show that this probability is maximum at $p^* = 1/12 = 0.08333$.
 (b) The maximum-likelihood estimate of p is the number p^* that maximizes

$$\binom{993}{299} p^{300} (1-p)^{694}.$$

As discussed in class, the maximum is at $p^* = 300/994 = 0.3018$.

- (c) The maximum-likelihood estimate of the unknown value of p is the number p^* that maximizes $\binom{1000}{300} p^{300} (1-p)^{700}$. We saw in class that the maximum is at $p^* = 300/1000 = 0.3$.
 (d) The difference in the two cases is that after having observed the 300th head on the 994th toss, we can estimate p^* as $300/994 = 0.3018$ *without knowing what happened on the next 6 trials*. If we are told that Tails occurred on the next 6 trials, we *do* receive additional information which causes us to be pessimistic about the value of p , and hence we revise the estimated value of p *downwards* to $300/1000 = 0.3$.
22. Define

$$X_i = \begin{cases} 1 & \text{if } A \text{ occurs on trial } i \\ 0 & \text{otherwise} \end{cases}$$

Random variable X_i is Bernoulli with success probability $P(A)$ so that $E[X_i] = P(A)$ and $\text{Var}(X_i) = P(A)(1 - P(A))$. For n experimental trials, we define the relative frequency of A as

$$R_n = \frac{X_1 + \dots + X_n}{n}$$

Since $E[R_n] = P(A)$, using the Chebyshev inequality for the random variable R_n we obtain

$$P\{|R_n - P(A)| \geq c\} \leq \frac{P(A)(1 - P(A))}{nc^2}$$

where c is the *confidence coefficient* and $2c$ is the *confidence interval*. Since we are given the *true value* of $P(A) = 0.01$, we can write

$$P\{|R_n - P(A)| \geq c\} \leq \frac{0.0099}{nc^2}$$

We meet the requirement by choosing $c = 0.001$ yielding

$$P\{|R_n - P(A)| \geq 0.001\} \leq \frac{9900}{n}$$

Thus to have confidence level 0.01, we require that $9900/n \leq 0.01$. This requires $n \geq 990,000$.