

8. (a) $P\{\text{divisible by } 3\} = \frac{333}{1000}$
 $P\{\text{divisible by } 7\} = \frac{142}{1000}$
 $P\{\text{divisible by } 15\} = \frac{66}{1000}$
 $P\{\text{divisible by } 105\} = \frac{9}{1000}$

In the limiting case, the probabilities converge to $1/3$, $1/7$, $1/15$ and $1/105$, respectively.

(b)

$$\begin{aligned} P\{\mu(N) = 0\} &= P\{N \text{ not divisible by } p_i^2, i \geq 2\} \\ &= \prod_i P\{N \text{ not divisible by } p_i^2\} \\ &= \prod_i (1 - 1/p_i^2) = 6/\pi^2 \end{aligned}$$

9. Let N denote the number of games played.

- (a) The game is played either 2 or 3 times. All possibilities are listed below
 A wins in 2 games with probability p^2
 B wins in 2 games with probability $(1-p)^2$
 A wins in 3 games with probability $2(1-p)p^2$
 B wins in 3 games with probability $2p(1-p)^2$
Therefore,

$$E(N) = 2[p^2 + (1-p)^2] + 3[2(1-p)p^2 + 2p(1-p)^2] = 2 + 2p(1-p)$$

Differentiation yields that

$$\frac{d}{dp} E[N] = 2 - 4p$$

and so the minimum occurs when $2 - 4p = 0$ or $p = 1/2$.

- (b) This time the game is played either 3, 4, or 5 times. The possibilities are
 A wins in 3 games with probability p^3
 B wins in 3 games with probability $(1-p)^3$
 A wins in 4 games with probability $3(1-p)p^3$
 B wins in 4 games with probability $3p(1-p)^3$
 A wins in 5 games with probability $6(1-p)^2p^3$
 B wins in 5 games with probability $6p^2(1-p)^3$
Hence,

$$E(N) = 3[p^3 + (1-p)^3] + 4[3(1-p)p^3 + 3p(1-p)^3] + 5[6(1-p)^2p^3 + 6p^2(1-p)^3]$$

Differentiation yields

$$\frac{d}{dp} E[N] = 24p^3 - 36p^2 + 6p + 3.$$

Its value at $p = 1/2$ is easily seen to be 0. Therefore, the minimum occurs when $p = 1/2$.

10.

$$E[X] = \sum_{n=1}^{\infty} 2^n (1/2)^n = \infty$$

- (a) Probably not.
(b) Yes, if you could play an arbitrary large number of games.

11. (a)

$$E[(2+X)^2] = \text{Var}(2+X) + (E[(2+X)])^2 = \text{Var}(X) + 9 = 14$$

(b)

$$\text{Var}(4+3X) = 9\text{Var}(X) = 45$$

12. (a) Since the balls are selected without replacement, the maximum of them is at least n . Hence, $P\{Y = j\} = 0$ if $1 \leq j \leq n-1$. Otherwise, the maximum is j when one of the balls has number j , and the remaining $n-1$ balls have numbers between 1 and $j-1$. Therefore,

$$P\{Y = j\} = \frac{\binom{j-1}{n-1}}{\binom{N}{n}}, \quad n \leq j \leq N$$

(b)

$$\begin{aligned} E[Y] &= \sum_{j=n}^N j \frac{\binom{j-1}{n-1}}{\binom{N}{n}} \\ &= \frac{n}{\binom{N}{n}} \sum_{j=n}^N \binom{j}{n} \\ &= \frac{n}{\binom{N}{n}} \sum_{i=n+1}^{N+1} \binom{i-1}{n+1-1} \\ &= \frac{n}{\binom{N}{n}} \binom{N+1}{n+1} \\ &= \frac{n(N+1)}{n+1} \end{aligned}$$

13. The boy draws k chips when the first $k-1$ chips are all distinct, and the k^{th} chip is equal to any one of the first $k-1$ chips. Therefore,

$$P\{X = k\} = \frac{k-1}{n} \prod_{i=0}^{k-2} \frac{n-i}{n}, \quad k > 1$$

14. (a) Let T denote the number of draws made before the red ball is drawn. Then T is a geometric random variable with parameter $\frac{1}{n}$ and therefore $E[T] = n$.

- (b) Let T_i be the random variable describing the number of draws required to get a new ball when i balls are already drawn and hence are old. Then, it is true that

$$E[T] = E\left[\sum_{i=0}^{n-1} T_i\right] = \sum_{i=0}^{n-1} E[T_i]$$

To compute $E[T_i]$, we observe that with probability $\frac{i}{n}$, an old ball is drawn and with probability $\frac{n-i}{n}$, it is new. T_i is a geometric random variable with parameter $\frac{n-i}{n}$. Hence, we have $E[T_i] = \frac{n}{n-i}$. Substituting this result in the above equation yields

$$E[T] = \sum_{i=0}^{n-1} \frac{n}{n-i} = n \sum_{i=1}^n \frac{1}{i}.$$

15. Let $E[X]$ denote the expected number of steps to freedom. Then we can write the following recursion

$$E[X] = \frac{1}{3} \cdot 9 + \frac{1}{3}(3 + E[X]) + \frac{1}{3}(6 + E[X]),$$

which yields $E[X] = 18$.