

Jointly Distributed Random Variables

Assigned reading: *Ross* Chapter 6, Sections 1 and 4.

Noncredit Exercises: pp. 290-295: 1-8, 10-12, 35-41.

1. The joint pmf of X and Y is

$$p_{X,Y}(i, j) = \begin{cases} 2^{-(i-1)}3^{-j} & : i = 1, 2, \dots; j = 1, 2, \dots \\ 0 & : \text{otherwise} \end{cases}$$

- (a) Find the marginals of X and Y . What kinds of random variables are these?

$$p_x(i) = \sum_{j=1}^{\infty} 2^{-(i-1)}3^{-j} = 2^{-(i-1)} \sum_{j=1}^{\infty} 3^{-j} = 2^{-(i-1)} \frac{1/3}{1-1/3} = 2^{-(i-1)} \frac{1}{2} = \left(1 - \frac{1}{2}\right) \left(\frac{1}{2}\right)^{i-1}, \quad i = 1, 2, \dots$$

This is a geometric random variable with parameter $p = 1/2$.

$$p_y(j) = \sum_{i=1}^{\infty} 2^{-(i-1)}3^{-j} = 3^{-j} \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^{i-1} = 3^{-j} \frac{1}{1-1/2} = 2 \cdot \left(\frac{1}{3}\right)^{-j} = \left(1 - \frac{1}{3}\right) \left(\frac{1}{3}\right)^{j-1}$$

This is a geometric random variable with parameter $p = 1/3$.

- (b) Find the conditional pmf of X given that $Y = k$.

The conditional pmf of X given that $Y = k$ is

$$p_{X|Y}(i|k) = \frac{\frac{1}{2}^{(i-1)} \frac{1}{3}^k}{\left(1 - \frac{1}{3}\right) \left(\frac{1}{3}\right)^{k-1}} = \left(\frac{1}{2}\right)^{i-1} \frac{1}{2} = p_x(i)$$

- (c) Are X and Y independent? Can you arrive at this conclusion from your answer to part (b)?

X and Y are independent. This is evident in part (a), since the joint pmf is equal to the product of the two marginal pmf's, and it is evident in part (b) since the conditional pmf for X given Y is equal to the marginal pmf for X .

2. The discrete random variables X and Y have the joint pmf $p_{X,Y}(u, v)$ given by the following table.

p_{ij}	u_j				
	0	1	3	5	
v_i	4	0	1/12	1/6	1/12
	3	1/6	1/12	0	1/12
	-1	1/12	1/6	1/12	0

For example, $p_{X,Y}(0, 3) = 1/6$.

- (a) Find the marginal pmfs $p_X(u)$ and $p_Y(v)$ of X and Y .

The marginal for X is given by

$$p_x(0) = 1/6 + 1/12 = 1/4, \quad p_x(1) = 1/12 + 1/12 + 1/6 = 1/3, \quad p_x(3) = 1/6 + 1/12 = 1/4, \quad p_x(5) = 1/12 + 1/12 = 1/6.$$

The marginal for Y is given by

$$p_y(4) = 1/12 + 1/6 + 1/12 = 1/3, \quad p_y(3) = 1/6 + 1/12 + 1/12 = 1/3, \quad p_y(-1) = 1/12 + 1/6 + 1/12 = 1/3.$$

- (b) Are the random variables X and Y independent.

Since $p_x(3)p_y(3) = 1/3 \times 1/4 = 1/12 \neq 0 = p_{xy}(3, 3)$, the random variables X and Y are not independent. This can also be determined (fairly quickly) because the non-zero probability masses are not on an axis-aligned grid.

- (c) Find $P\{X \leq Y\}$ and $P\{X + Y \leq 8\}$.

$$P\{X \leq Y\} = p_{xy}(0, 3) + p_{xy}(1, 4) + p_{xy}(1, 3) + p_{xy}(3, 4) = 1/6 + 1/12 + 1/12 + 1/6 = 1/2$$

$$P\{X + Y \leq 8\} = 1 - P\{X + Y > 8\} = 1 - p_{xy}(5, 4) = 11/12$$

3. The jointly continuous random variables X and Y have joint pdf given by

$$f_{X,Y}(u, v) = \begin{cases} 2e^{-(u+v)} & : 0 < u < v < \infty \\ 0 & : \text{otherwise} \end{cases}$$

- (a) Sketch the $u - v$ plane and indicate on it the region over which $f_{X,Y}(u, v)$ is nonzero.

The joint pdf is non-zero in the positive quadrant, above the line $u = v$. Although they are not shown in these solutions, it is very useful to sketch this region and the regions of integration for each of the parts to this problem.

- (b) Find the marginal pdfs of X and Y .

$$f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u, v) dv = 2e^{-u} \int_u^{\infty} e^{-v} dv = 2e^{-u} \left(-e^{-v} \Big|_u^{\infty} \right) = 2e^{-2u}$$

$$f_Y(u) = \int_{-\infty}^{\infty} f_{X,Y}(u, v) du = 2e^{-v} \int_0^v e^{-u} du = 2e^{-v} \left(-e^{-u} \Big|_0^v \right) = 2e^{-v} (-e^{-v} + 1) = 2e^{-v} - 2e^{-2v}$$

- (c) Find $P\{Y > 3X\}$.

Here, we integrate above the line $v = 3u$, with u ranging from 0 to ∞ .

$$P\{Y > 3X\} = \int_0^{\infty} \int_{3u}^{\infty} 2e^{-(u+v)} dv du = 2 \int_0^{\infty} e^{-u} \int_{3u}^{\infty} e^{-v} dv du = 2 \int_0^{\infty} e^{-4u} du = 2 \left(-\frac{1}{4} e^{-4u} \Big|_0^{\infty} \right) = \frac{1}{2}$$

- (d) For $a > 0$, find $P\{X + Y \leq a\}$.

Here, a sketch will show that u ranges from 0 to $a/2$, while for a specific value of u , v can take values from u to $a - u$.

$$\begin{aligned} P\{X + Y \leq a\} &= \int_0^{\frac{a}{2}} \int_u^{a-u} 2e^{-(u+v)} dv du = 2 \int_0^{\frac{a}{2}} e^{-u} \int_u^{a-u} e^{-v} dv du = 2 \int_0^{\frac{a}{2}} e^{-u} (-e^{-(a-u)} + e^{-u}) du \\ &= 2 \int_0^{\frac{a}{2}} (-e^{-a} + e^{-2u}) du = 2 \int_0^{\frac{a}{2}} e^{-2u} du - 2 \int_0^{\frac{a}{2}} e^{-a} du = -e^{-a} + 1 - ae^{-a} = 1 - e^{-a}(1 + a) \end{aligned}$$

4. The jointly continuous random variables X and Y have joint pdf given by

$$f_{X,Y}(u, v) = \begin{cases} 1/2 & : 0 \leq u < 1, 0 \leq v < 1, 0 \leq u + v < 1 \\ 3/2 & : 0 \leq u < 1, 0 \leq v < 1, 1 \leq u + v < 2 \\ 0 & : \text{otherwise} \end{cases}$$

- (a) Find the marginal pdf of X .

In the unit square (with corners $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$), the pdf value is $1/2$ for the lower left triangle (i.e., the triangle with coordinates $(0,0)$, $(0,1)$, $(1,0)$) and $3/2$ for the upper right triangle, (i.e., the triangle with coordinates $(1,1)$, $(0,1)$, $(1,0)$). To find the marginal for X ,

$$f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u, v) dv = \int_0^{1-u} \frac{1}{2} dv + \int_{1-u}^1 \frac{3}{2} dv = \frac{1}{2}(1-u) + \frac{3}{2}u = \frac{1}{2} + u, \quad 0 \leq u \leq 1$$

- (b) Find $P\{X + Y \leq 3/2\}$.

It is easy to see that $P\{X + Y > 3/2\}$ is equal to the area of the triangle with sides of length $1/2$ times $3/2$ (which is the density value in the upper right corner of the unit square).

$$P\{X + Y \leq 3/2\} = 1 - P\{X + Y > 3/2\} = 1 - \frac{3}{2} \int_{\frac{1}{2}}^1 \int_{\frac{3}{2}-u}^1 dv du = 1 - \frac{3}{2} \int_{\frac{1}{2}}^1 (u - \frac{1}{2}) du = 1 - \frac{3}{2} \left(\frac{u^2}{2} - \frac{u}{2} \Big|_{\frac{1}{2}}^1 \right) = 1 - \frac{3}{16} = \frac{13}{16}$$

- (c) Find $P\{X^2 + Y^2 \geq 1\}$.

The easy way to solve this problem is to note the region of interest is the area contained in the unit square, but outside of the unit circle. The area of this region is simply $1 - \pi/4$, and the pdf value is $3/2$ over this region, giving $P\{X^2 + Y^2 \geq 1\} = 3/2(1 - \pi/4)$. Of course, one can always integrate...

$$\begin{aligned} P\{X^2 + Y^2 \geq 1\} &= \frac{3}{2} \int_0^1 \int_{\sqrt{1-u^2}}^1 dv du = \frac{3}{2} \int_0^1 (1 - \sqrt{1-u^2}) du = \frac{3}{2} \left(u - \frac{u}{2} \sqrt{1-u^2} - \frac{1}{2} \sin^{-1} u \right) \Big|_0^1 \\ &= \frac{3}{2} \left(1 - \frac{1}{2} \sin^{-1} 1 + \frac{1}{2} \sin^{-1} 0 \right) = \frac{3}{2} \left(1 - \frac{\pi}{4} \right) = \frac{3}{2} - \frac{3\pi}{8} \end{aligned}$$

5. Problem 9, pg. 290 of *Ross*.

- (a) $f_{XY} \geq 0$ and $\int f_{XY}(u, v) du dv = 1$

(b)

$$f_X(u) = \frac{6}{7} \int_0^2 \left(u^2 + \frac{uv}{2} \right) dv = \frac{6}{7} (2u^2 + u)$$

(c)

$$P\{\mathbf{X} > \mathbf{Y}\} = \frac{6}{7} \int_0^1 \int_0^u \left(u^2 + \frac{uv}{2} \right) dv du = \frac{15}{56}$$

(d)

$$P\{\mathbf{Y} > \frac{1}{2} \mid \mathbf{X} < \frac{1}{2}\} = \frac{P\{\mathbf{Y} > \frac{1}{2}, \mathbf{X} < \frac{1}{2}\}}{P\{\mathbf{X} < \frac{1}{2}\}} = \frac{\int_{\frac{1}{2}}^2 \int_0^{\frac{1}{2}} \left(u^2 + \frac{uv}{2} \right) du dv}{\int_0^{\frac{1}{2}} (2u^2 + u) du} = \frac{69}{80} = 0.8625$$

6. The number of α -particles emitted by a source during a unit time interval can be modeled as a Poisson random variable X with parameter λ . The α -particles are detected by means of a Geiger counter which detects a particle with probability $p < 1$ (i.e., the Geiger counter is not 100% accurate). The detections of the particles can be considered independent events. Thus, if n particles have been emitted, the Geiger counter reading can be modeled as a binomial random variable Y with parameters (n, p) . Thus, the conditional pmf of Y given that $X = n$ is a binomial pmf

$$p_{Y|X}(k|n) = \binom{n}{k} p^k (1-p)^{n-k}, \quad 0 \leq k \leq n.$$

Note that $Y \leq X$, i.e., the counter never mistakenly counts a particle that was not emitted.

- (a) Sketch the $u-v$ plane and the joint pmf of X and Y . Precision is not required in the sizes of the blobs that you draw to represent probability mass, but be sure that you do not indicate nonzero probability mass where there should be no probability mass.

Since $Y \leq X$, non-zero probability mass is assigned to those point with non-negative integer coordinates such that these points lie below the line $u = v$.

- (b) What is the unconditional pmf of Y ?

We can use the total probability to compute $p_Y(k)$ (NOTE: $p_{Y|X}(k|n) = 0$ for $n < k$):

$$\begin{aligned} p_Y(k) &= \sum_{n=k}^{\infty} p_{Y|X}(k|n) p_X(n) \\ &= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{\lambda^n}{n!} e^{-\lambda} = e^{-\lambda} p^k \lambda^k \sum_{n=k}^{\infty} \binom{n}{k} (1-p)^{n-k} \frac{\lambda^{n-k}}{n!} \\ &= \frac{(\lambda p)^k}{k!} e^{-\lambda} \sum_{n=k}^{\infty} \frac{[(1-p)\lambda]^{n-k}}{(n-k)!} = \frac{(\lambda p)^k}{k!} e^{-\lambda} \sum_{l=0}^{\infty} \frac{[(1-p)\lambda]^l}{(l)!} = \frac{(\lambda p)^k}{k!} e^{-\lambda} e^{[(1-p)\lambda]} = \frac{(\lambda p)^k}{k!} e^{-\lambda p}, \quad k \geq 0 \end{aligned}$$

So, Y is Poisson with parameter λp .

(c) What is the conditional pmf of X given $Y = k$?

We compute the conditional pmf of X given $Y = k$ using Bayes theorem.

$$\begin{aligned} p_{X|Y}(n|k) &= \frac{p_{Y|X}(k|n)p_X(n)}{p_Y(k)} = \frac{\binom{n}{k} p^k (1-p)^{n-k} \frac{\lambda^n}{n!} e^{-\lambda}}{\frac{(\lambda p)^k}{k!} e^{-\lambda p}} = \binom{n}{k} p^k (1-p)^{n-k} \frac{\lambda^n}{n!} e^{-\lambda} \frac{k!}{(\lambda p)^k e^{-\lambda p}} \\ &= \frac{(1-p)^{n-k}}{(n-k)!} \frac{\lambda^n e^{-\lambda}}{(\lambda)^k e^{-\lambda p}} = \frac{[\lambda(1-p)]^{n-k}}{(n-k)!} e^{-\lambda(1-p)} \end{aligned}$$

This is the pmf for a Poisson random variable with parameter $\lambda(1-p)$, shifted to the right by k .