

1. Generation of random variables with specified probability density function

By integration, $F_X(x) = x^2$ for $0 \leq x \leq 1$. It is desired that $P[g(U) \leq x] = x^2$. For fixed x , let u be the value so that $g(u) = x$. The event $\{g(U) \leq x\}$ is thus equivalent to $\{U \leq u\}$, which has probability u . Thus, we want $u = x^2$, or equivalently, $x = \sqrt{u}$. That is, $g(u) = \sqrt{u}$ for $0 \leq u \leq 1$ does the trick.

(Note: In general, for each x , the cumulative probability distribution function desired for $g(U)$ determines value u such that $g(u) = x$. In other words, the inverse of the function g is specified, from which we can find g .)

2. Normal approximation to the binomial distribution

The number of people S favoring the proposition is a binomial random variable with parameters $(100, 0.65)$. It has mean $\mu = 65$ and variance $\sigma^2 = 100(0.65)(1 - 0.65) = 22.75$. The DeMoivre-Laplace limit theorem tells us that S is well approximated by a normal random variable with parameters μ and σ^2 .

$$(a) P[S \geq 50] = P\left[\frac{S-65}{\sqrt{22.75}} \geq \frac{50-65}{\sqrt{22.75}}\right] \approx 1 - \Phi(-3.14) = \Phi(3.14) = 0.9992$$

$$(b) P[60 \leq S \leq 70] = P\left[\frac{60-65}{\sqrt{22.75}} \leq \frac{S-65}{\sqrt{22.75}} \leq \frac{70-65}{\sqrt{22.75}}\right] \approx \Phi(1.048) - \Phi(-1.048) = 2\Phi(1.048) - 1 = 0.706$$

$$(c) P[S \leq 75] = P\left[\frac{S-65}{\sqrt{22.75}} \leq \frac{75-65}{\sqrt{22.75}}\right] \approx \Phi(2.096) = 0.9778$$

3. The Poisson process

$$(a) P[N = n] = \frac{e^{-\mu T} (\mu T)^n}{n!}$$

$$(b) P[N_1 = i] = \frac{e^{-\mu \tau} (\mu \tau)^i}{i!}$$

$$(c) P[N_2 = j] = \frac{e^{-\mu(T-\tau)} (\mu(T-\tau))^j}{j!}$$

(d)

$$P[N_1 = i | N = n] = \frac{P[N_1 = i, N = n]}{P[N = n]} = \frac{P[N_1 = i, N_2 = j]}{P[N = n]} = \frac{P[N_1 = i]P[N_2 = j]}{P[N = n]} = \frac{n!}{i!j!} \left(\frac{\tau}{T}\right)^i \left(\frac{T-\tau}{T}\right)^j$$

which can also be written as $P[N_1 = i | N = n] = \binom{n}{i} p^i (1-p)^{n-i}$, where $p = \frac{\tau}{T}$. As a function of i , the answer is the pmf of a binomial probability.

(e)

$$P[N = n | N_1 = i] = \frac{P[N = n, N_1 = i]}{P[N_1 = i]} = \frac{P[N_1 = i, N_2 = j]}{P[N_1 = i]} = \frac{P[N_1 = i]P[N_2 = j]}{P[N_1 = i]} = P[N_2 = j] = \frac{e^{-\mu(T-\tau)} (\mu(T-\tau))^j}{j!}$$

which can also be written as $P[N = n | N_1 = i] = \frac{e^{-\mu(T-\tau)} (\mu(T-\tau))^{n-i}}{(n-i)!}$. That is, given $N_1 = i$, the total number of arrivals is i plus a random number of arrivals. The random number of arrivals has the Poisson distribution with mean $\mu(T-\tau)$.

4. Waiting times for a Poisson process

$$(a) P[N(t) = i] = \frac{e^{-2t} (2t)^i}{i!} \text{ for nonnegative integers } i \text{ (Poisson distribution, mean } 2t)$$

$$(b) P[E_1] = P[N(3.5) = 0] = \frac{e^{-7} (7)^0}{0!} = e^{-7} = 0.00091$$

$P[E_2] = 0.00091$ since E_2 is the same event as E_1

$$P[E_3] = P[N(3.5) \leq 2] = P[N(3.5) = 0] + P[N(3.5) = 1] + P[N(3.5) = 2] = e^{-7} (1 + 7 + \frac{7^2}{2}) = 0.0296$$

$P[E_4] = 0.0296$ since E_4 is the same event as E_3

$$P[E_5] = P[N(t) \leq 2] = P[N(t) = 0] + P[N(t) = 1] + P[N(t) = 2] = e^{-2t} (1 + 2t + \frac{(2t)^2}{2})$$

$$P[E_6] = 1 - P[E_5] = 1 - e^{-2t} (1 + 2t + \frac{(2t)^2}{2})$$

(c) Differentiating yields $f(t) = e^{-2t} (2(1 + 2t + \frac{(2t)^2}{2}) - 2 - 4t) = e^{-2t} \frac{(2t)^2}{2}$ which indeed is the gamma density with parameters $(i = 3, \mu = 2)$.

(d) The expected time between arrivals is $1/\mu$, so the expected time until the 10^{th} arrival is $10/\mu = 5$ minutes.

5. Hypothesis testing for the variance of a Gaussian variate

(a)

$$\Lambda(x) = \frac{\frac{1}{b\sqrt{2\pi}}e^{-\frac{x^2}{2b^2}}}{\frac{1}{a\sqrt{2\pi}}e^{-\frac{x^2}{2a^2}}} = \frac{a}{b}e^{-\frac{x^2}{2b^2} + \frac{x^2}{2a^2}} = \frac{a}{b}e^{\frac{x^2}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)}$$

(b) The ML rule is to choose H_1 when $\Lambda(X) > 1$. Thus, by taking the natural logarithm of both sides of this inequality we obtain the rule: If $(\ln \frac{a}{b}) + \frac{X^2}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) > 0$ choose H_1 . After a bit of algebra, we derive the rule that H_1 should be chosen when

$$\ln \frac{b}{a} < \frac{X^2}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{X^2}{2} \frac{b^2 - a^2}{a^2 b^2} \Rightarrow \left(\frac{2a^2 b^2}{b^2 - a^2}\right) \left(\ln \frac{b}{a}\right) < X^2 \Rightarrow K = ab\sqrt{\frac{2(\ln b - \ln a)}{b^2 - a^2}} < |X|$$

(c) The MAP rule is to choose H_1 when $\Lambda(X) > \frac{\pi_0}{\pi_1}$. After a bit of algebra, we derive the rule that H_1 should be chosen when

$$\ln \frac{b\pi_0}{a\pi_1} < \frac{X^2}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) = \frac{X^2}{2} \frac{b^2 - a^2}{a^2 b^2} \Rightarrow K = ab\sqrt{\frac{2(\ln(b\pi_0) - \ln(a\pi_1))}{b^2 - a^2}} < |X|$$

(d) $p_{false_alarm} = P\{|X| > K \mid H_0\} = \int_{-\infty}^{-K} f_0(u)du + \int_K^{\infty} f_0(u)du = \Phi(-K/a) + 1 - \Phi(K/a) = 2Q(K/a)$.
 $p_{miss} = P\{|X| < K \mid H_1\} = \int_{-K}^K f_1(u)du = \Phi(K/b) - \Phi(-K/b) = 1 - 2Q(K/b)$.

6. Hypothesis testing for the mean of a Laplace variate

(a) It is helpful to explicitly note that

$$|u - 1| = \begin{cases} -u + 1 & : u < 1 \\ u - 1 & : u \geq 1 \end{cases}, \quad |u + 1| = \begin{cases} -u - 1 & : u < -1 \\ u + 1 & : u \geq -1 \end{cases}.$$

Thus,

$$f_1(u) = \begin{cases} \frac{1}{2}e^{u-1} & : u < 1 \\ \frac{1}{2}e^{-u+1} & : u \geq 1 \end{cases} \quad f_0(u) = \begin{cases} \frac{1}{2}e^{u+1} & : u < -1 \\ \frac{1}{2}e^{-u-1} & : u \geq -1 \end{cases}$$

(b)

$$\Lambda(u) = \frac{e^{-|u-1|}}{e^{-|u+1|}} = \begin{cases} \frac{e^{u-1}}{e^{u+1}} = e^{-2} & : u < -1 \\ \frac{e^{u-1}}{e^{-u-1}} = e^{2u} & : -1 \leq u < 1 \\ \frac{e^{-u+1}}{e^{-u-1}} = e^2 & : 1 < u \end{cases}$$

(c) The likelihood ratio $\Lambda(u)$ is nondecreasing and it crosses 1 at $u = 0$. Thus, the ML decision rule is to decide H_1 is true if $X > 0$ and decide H_0 otherwise.

(d)

$$p_{false_alarm} = \int_0^{\infty} f_0(u)du = \int_0^{\infty} \frac{1}{2}e^{-u-1}du = \frac{1}{2}e^{-1} \int_0^{\infty} e^{-u}du = \frac{1}{2}e^{-1}$$

By symmetry, we see that p_{miss} is also given by $p_{miss} = \frac{1}{2}e^{-1}$ (of course, one could always compute the integral $\int_{-\infty}^0 \frac{1}{2}e^{-|u-1|}du = \int_{-\infty}^0 \frac{1}{2}e^{(u-1)}du = \frac{1}{2}e^{-1} \int_{-\infty}^0 \frac{1}{2}e^u du = \frac{1}{2}e^{-1}$).

(e) The MAP decision rule is to choose H_1 if $\Lambda(u) > \frac{\pi_0}{\pi_1} = 2$, and choose H_0 otherwise. Note that the solution of $e^{2u} = 2$ is $u = \frac{\ln 2}{2}$, which is between -1 and 1. Thus, the MAP decision rule is to choose H_1 if $X \geq K$ and choose H_0 otherwise, where $K = \frac{\ln 2}{2} = \ln \sqrt{2}$.

(f)

$$p_{false_alarm} = \int_{\ln \sqrt{2}}^{\infty} f_0(u)du = \frac{1}{2}e^{-1} \int_{\ln \sqrt{2}}^{\infty} e^{-u}du = \frac{1}{2}e^{-1}(-e^{-\infty} + e^{-\ln \sqrt{2}}) = \frac{1}{2\sqrt{2}}e^{-1}.$$

Similarly,

$$p_{miss} = \int_{-\infty}^{\ln \sqrt{2}} f_1(u)du = \frac{1}{2}e^{-1} \int_{-\infty}^{\ln \sqrt{2}} e^u du = \frac{1}{2}e^{-1}e^{\ln \sqrt{2}} = \sqrt{2}\frac{1}{2}e^{-1} = \frac{1}{\sqrt{2}}e^{-1}.$$