

1. Computing probabilities for Gaussian random variables.

Let X be a Gaussian random variable with mean $\mu = -1$ and variance $\sigma^2 = 4$. Let $\Phi(x)$ denote the CDF of a standard Gaussian random variable, and let $Q(x) = 1 - \Phi(x)$. You have at your disposal two calculators. Calculator A can evaluate only $\Phi(x)$, and only for nonnegative values of x , while Calculator B can evaluate only $Q(x)$, again only for nonnegative values of x . Both calculators perform the usual assortment of standard arithmetic functions. For each of the following probabilities, write down *two* alternative expressions: one for the evaluation using Calculator A and one for evaluation using calculator B.

- (a) $P(X < 0)$
- (b) $P(-10 < X < 5)$
- (c) $P(|X| \geq 5)$
- (d) $P(X^2 - 3X + 2 < 0)$

For this problem, we'll use the fact that $Z \sim \mathcal{N}(\mu, \sigma^2)$, we have $F_Z(u) = \Phi(\frac{u-\mu}{\sigma})$. Furthermore, we have $Q(x) = 1 - \Phi(x)$ and $\Phi(-x) = 1 - \Phi(x)$.

	Calculator A	Calculator B
$P(X < 0)$	$\Phi(0.5)$	$1 - Q(0.5)$
$P(-10 < X < 5)$	$\Phi(3) + \Phi(4.5) - 1$	$1 - Q(3) - Q(4.5)$
$P(X \geq 5)$	$1 - \Phi(3) + 1 - \Phi(2)$	$Q(3) + Q(2)$
$P(X^2 - 3X + 2 < 0)$	$\Phi(1.5) - \Phi(1)$	$Q(1) - Q(1.5)$

2. Quality Control

The width of a metal trace on a circuit board is a Gaussian random variable with mean $\mu = 0.9$ microns and standard deviation $\sigma = 0.003$ microns.

- (a) If the width of a metal trace is not within the limits 0.9 ± 0.005 microns, then the trace is deemed to be defective. What percentage of the traces will be defective?

Let $X \sim (0.9, 0.003^2)$ be the random variable that denotes trace width. The trace is defective if $\{X < 0.9 - 0.005\}$ or $\{X > 0.9 + 0.005\}$. Therefore, the probability that a trace is defective is just $P[\{X < 0.9 - 0.005\} \cup \{X > 0.9 + 0.005\}] = 2F_X(0.895) = \Phi(\frac{0.895-0.9}{0.003}) \approx 0.095$.

- (b) The CEO's goal is to make sure that, on average, no more than 1 in 100 traces are defective. What is the maximum allowable value of σ that will enable the company to achieve this goal?

Using the result of Part (a), we need to find σ such that $\Phi(\frac{0.895-0.9}{\sigma}) \leq 0.01$, i.e., $\Phi(0.005/\sigma) \geq 0.995$. Consulting the tables, we find that $\Phi(x) \geq .995$ for $x > 2.57$ which gives $\sigma \leq 0.00195$.

- (c) Suppose that $\mu = 0.9$ microns and $\sigma = 0.003$ microns (as before) but that, due to technological advances in the circuit board industry, the specification has been revised, and now traces whose widths do not fall within the range 0.896 microns to 0.908 microns are deemed to be defective. What percentage of the traces will be defective under this new specification?

Here, we need the probability

$$P[\{X < 0.896\} \cup \{X > 0.908\}] = 1 - [F_X(0.908) - F_X(0.896)] = 1 - [\Phi(2.67) - \Phi(-1.33)] \approx 0.0956.$$

3. Optimal Quantizers

An analog signal X is modeled as a standard Gaussian random variable. For some applications, only a quantized value is used (e.g., an A/D converter converts the analog signal to one of two possible digitized values). Let Y denote the quantized value, with $Y = \alpha$ for $X > 0$ and $Y = -\alpha$ for $X \leq 0$.

- (a) What is the pmf of Y ?

Since $X \sim \mathcal{N}(0, 1)$, we have $P\{X \leq 0\} = P\{X > 0\} = \frac{1}{2}$. Thus, $p_Y(\alpha) = p_Y(-\alpha) = \frac{1}{2}$

- (b) Suppose that $\alpha = 1$. If the signal X happens to have the value 1.29, what is the error made in representing X by Y ? What is the squared error, $Z = (X - Y)^2$? What are the error and squared errors for the cases of $X = \pi/4$ and $X = -\pi/4$?

The error is $1.29 - 1 = 0.29$ and the squared error is $(0.29)^2 = 0.0841$. For $\pi/4$ we have $\pi/4 - 1 = -0.215$ and the squared error is $(-0.215)^2 = 0.046$. For $-\pi/4$ we have $-\pi/4 - 1 = -0.215$ and the squared error is $(-0.215)^2 = 0.046$. Note that the errors are the same for $\pm\pi/4$.

- (c) You have been given the task of designing the quantizer that minimizes the expected value of the squared error, $E[Z]$. Your only design choice here is the value for α . Find $E[Z]$ as a function of α , and then find the value of α that minimizes $E[Z]$.

$$\begin{aligned}
 E[Z] &= \int_{-\infty}^0 (u + \alpha)^2 f(u) du + \int_0^{\infty} (u - \alpha)^2 f(u) du \\
 &= \int_{-\infty}^0 (u^2 + 2u\alpha + \alpha^2) f(u) du + \int_0^{\infty} (u^2 - 2u\alpha + \alpha^2) f(u) du \\
 &= \int_{-\infty}^{\infty} u^2 f(u) du + \int_{-\infty}^{\infty} \alpha^2 f(u) du + \int_{-\infty}^0 2u\alpha f(u) du - \int_0^{\infty} 2u\alpha f(u) du \\
 &= \int_{-\infty}^{\infty} u^2 f(u) du + \alpha^2 \int_{-\infty}^{\infty} f(u) du - \int_{-\infty}^0 2u\alpha f(u) du - \int_0^{\infty} 2u\alpha f(u) du \\
 &= E[X^2] + \alpha^2 - 4\alpha \int_0^{\infty} u f(u) du = 1 + \alpha^2 - \frac{4\alpha}{\sqrt{2\pi}}
 \end{aligned}$$

The third integral is solved by

$$\int_0^{\infty} u f(u) du = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} u e^{-\frac{u^2}{2}} du = -\frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \Big|_0^{\infty} = \frac{1}{\sqrt{2\pi}}$$

To minimize w.r.t. α , take the derivative w.r.t. α and set it to zero:

$$2\alpha = \frac{4}{\sqrt{2\pi}} \Rightarrow \alpha = \sqrt{\frac{2}{\pi}}.$$

- (d) In the new generation of products, a 3-bit A/D converter will be used. X will be quantized to the nearest integer W in the range -3 to 3 (e.g., $W = 3$ when $X \geq 2.5$, $W = 2$ when $1.5 \leq X < 2.5$, etc.). Find the pmf of W .

We can find the pmf for W by using the table of Φ values: $P\{W = -3\} = P\{W = 3\} = \Phi(-2.5) = 0.0062$, $P\{W = 0\} = \Phi(0.5) - \Phi(-0.5) = 0.3830$, $P\{W = -1\} = P\{W = 1\} = \Phi(1.5) - \Phi(0.5) = 0.2417$, $P\{W = -2\} = P\{W = 2\} = \Phi(2.5) - \Phi(1.5) = 0.0606$.

- (e) The output of the A/D converter is a 3-bit two's complement representation of W . Suppose that the output is (Z_2, Z_1, Z_0) , in which Z_0 the least significant bit. What are the pmf's for Z_2 , Z_1 , and Z_0 ?

$$P\{Z_2 = 1\} = P\{W < 0\} = P\{W = -3\} + P\{W = -2\} + P\{W = -1\} = 0.3085$$

$$P\{Z_1 = 1\} = P\{W = 2\} + P\{W = 3\} + P\{W = -1\} + P\{W = -2\} = 0.03691.$$

$$P\{Z_0 = 1\} = P\{W = 1\} + P\{W = 2\} + P\{W = 0\} + P\{W = -2\} = 0.05042.$$

4. System Reliability

The lifetime of a system with hazard rate $\lambda(t) = bt$ is a Rayleigh random variable X with pdf $f(u) = (bu)e^{-\frac{bu^2}{2}}$ for $u > 0$.

- (a) What is the *median* lifetime of the system? (Recall that the median is the value u s.t. $F_X(u) = 1/2$.) Is the median larger or smaller than the mean lifetime? How do these compare to the *mode* of the lifetime?

The median lifetime is the value T that satisfies

$$\frac{1}{2} = P\{X > T\} = \int_T^{\infty} (bu)e^{-\frac{bu^2}{2}} du \Rightarrow -e^{-\frac{bu^2}{2}} \Big|_T^{\infty} = e^{-\frac{bT^2}{2}} = \frac{1}{2} \Rightarrow T = \sqrt{2 \ln 2} \sqrt{1/b}$$

Using the substitution $v = bu^2/2$, the mean for a Rayleigh random variable is given by

$$E[X] = \int_0^\infty (bu)e^{-\frac{bu^2}{2}} du = \sqrt{2/b} \int_0^\infty \sqrt{v}e^{-v} dv.$$

But the last integral is just the gamma function $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi/4}$, which gives $E[X] = \sqrt{2/b}\sqrt{\pi/4} = \sqrt{1/b}\sqrt{\pi/2}$. (NOTE: showing that $\Gamma(1/2) = \sqrt{\pi}$ is exercise 20, pg. 239 of Ross.)

The mode is easily found to be at $u = \sqrt{1/b}$. Since $\sqrt{1/b} < \sqrt{2\ln 2}\sqrt{1/b} < \sqrt{\pi/2}\sqrt{1/b}$, we see that the mode is less than the median is less than the mean.

- (b) It is observed that the system fails at time $X = t$. What is the maximum-likelihood estimate of the parameter b ? (Note: the maximum-likelihood estimate of b is the value of b that maximizes the likelihood $f(t)$.)

$$0 = \frac{d}{db}(bt)e^{-\frac{bt^2}{2}} = te^{-\frac{bt^2}{2}} - \frac{t^2}{2}bte^{-\frac{bt^2}{2}} = te^{-\frac{bt^2}{2}}(1 - \frac{t^2}{2}b) \Rightarrow b = \frac{2}{t^2}$$

5. Functions of a random variable I.

Let the random variable X be uniform on the unit interval $(0, 1)$. Let $Y = -\ln X$. Find the pdf/pmf of Y .

Clearly Y is a continuous random variable, and therefore we will find its pdf (not a pmf). The random variable Y takes all values in the interval $(0, \infty)$. Now, to find the pdf for Y ,

$$F_Y(v) = P\{Y \leq v\} = P\{-\ln X \leq v\} = P\{X > e^{-v}\} = \int_{e^{-v}}^1 dx = 1 - e^{-v}, \quad v > 0$$

To find the pdf, simply differentiate w.r.t. v (for $v > 0$):

$$\frac{d}{dv}(1 - e^{-v}) \Rightarrow f_Y(v) = \begin{cases} e^{-v} & : v > 0 \\ 0 & : \text{otherwise} \end{cases}$$

6. Functions of a random variable II.

Let X be a random variable with pdf given by $f_X(u) = \frac{1}{2}u^{-2}$ for $|u| \geq 1$, and zero otherwise.

- (a) Determine the pdf/pmf of Y when $Y = \text{sign}(X + 2)$ with

$$\text{sign}(x) = \begin{cases} 1 & : x \geq 0 \\ -1 & : x < 0 \end{cases}.$$

Clearly, Y is a discrete random variable (it can take only the two values $\{-1, 1\}$), so we will determine its pmf.

$$p_Y(-1) = P\{Y = -1\} = P\{X + 2 < 0\} = P\{X < -2\} = \frac{1}{2} \int_{-\infty}^{-2} u^{-2} du = -\frac{1}{2}u^{-1} \Big|_{-\infty}^{-2} = \frac{1}{4}$$

and

$$\begin{aligned} p_Y(1) = P\{Y = 1\} &= P\{X + 2 \geq 0\} = P\{X \geq -2\} \\ &= \frac{1}{2} \int_{-2}^{-1} u^{-2} du + \frac{1}{2} \int_1^{\infty} u^{-2} du = -\frac{1}{2}u^{-1} \Big|_{-2}^{-1} - \frac{1}{2}u^{-1} \Big|_1^{\infty} \\ &= \frac{1}{2} - \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \end{aligned}$$

Of course the easier way to determine $p_Y(1)$ is to use $p_Y(1) = 1 - p_Y(-1)$.

- (b) Find the pmf/pdf for Z when Z is defined as

$$Z = \begin{cases} 2|X| & : |X| \leq 2 \\ X^2 & : |X| > 2 \end{cases}.$$

To find the pdf for Z , we'll find the CDF $F_Z(v)$ and differentiate. First, note that Z cannot be negative regardless of the value of X , and since $|X| > 1$, we see that $Z \geq 2$. Thus, $F_Z(v) = 0$ for all $v < 2$. For $1 \leq |X| \leq 2$, we have $Z = 2|X|$ and thus $2 \leq Z \leq 4$. So, for $2 \leq v \leq 4$ we have

$$\begin{aligned}
F_Z(v) &= P\{Z \leq v\} = P\{2|X| \leq v\} = P\{-\frac{v}{2} \leq X \leq \frac{v}{2}\} \\
&= \frac{1}{2} \int_{-\frac{v}{2}}^{-1} u^{-2} du + \frac{1}{2} \int_1^{\frac{v}{2}} u^{-2} du = -\frac{1}{2} u^{-1} \Big|_{-\frac{v}{2}}^{-1} - \frac{1}{2} u^{-1} \Big|_1^{\frac{v}{2}} \\
&= \left(\frac{1}{2} - \frac{1}{v}\right) - \left(\frac{1}{v} - \frac{1}{2}\right) = 1 - \frac{2}{v}
\end{aligned}$$

Finally, for $|X| > 2$ we have $Z = X^2$ and thus, for $v > 4$ we obtain

$$\begin{aligned}
F_Z(v) &= P\{Z \leq v\} = P\{X^2 \leq v\} = P\{X \leq \sqrt{v}\} \\
&= \frac{1}{2} \int_2^{\sqrt{v}} u^{-2} du = -\frac{1}{2} u^{-1} \Big|_2^{\sqrt{v}} = -\frac{1}{2\sqrt{v}} + \frac{1}{4}
\end{aligned}$$

Putting this all together, we obtain

$$F_Z(v) = \begin{cases} 0 & : v < 2 \\ 1 - \frac{2}{v} & : 2 \leq v \leq 4 \\ -\frac{1}{2\sqrt{v}} + \frac{1}{4} & : v > 4 \end{cases}$$

To obtain the pdf for Z , simply differentiate the CDF, giving

$$f_Z(v) = \begin{cases} 0 & : v < 2 \\ \frac{2}{v^2} & : 2 \leq v \leq 4 \\ \frac{1}{4} v^{-\frac{3}{2}} & : v > 4 \end{cases}$$

An alternative solution is to compute F_Z directly in terms of F_X . Again, for $v < 2$ we have $F_Z(v) = 0$. For $1 \leq |X| \leq 2$, we have $Z = 2|X|$ and thus $2 \leq Z \leq 4$. So, for $2 \leq v \leq 4$ we have

$$F_Z(v) = P\{Z \leq v\} = P\{2|X| \leq v\} = P\{-\frac{v}{2} \leq X \leq \frac{v}{2}\} = F_X(\frac{v}{2}) - F_X(-\frac{v}{2}).$$

For $|X| > 2$ we have $Z = X^2$ and thus, for $v > 4$ we obtain

$$F_Z(v) = P\{Z \leq v\} = P\{X^2 \leq v\} = P\{X \leq \sqrt{v}\} = F_X(\sqrt{v}).$$

and, thus

$$F_Z(v) = \begin{cases} 0 & : v < 2 \\ F_X(\frac{v}{2}) - F_X(-\frac{v}{2}) & : 2 \leq v \leq 4 \\ F_X(\sqrt{v}) & : v > 4 \end{cases}$$

Finally, to compute the pdf for Z , differentiate to obtain

$$f_Z(v) = \begin{cases} 0 & : v < 2 \\ \frac{1}{2} f_X(\frac{v}{2}) + \frac{1}{2} f_X(-\frac{v}{2}) & : 2 \leq v \leq 4 \\ \frac{1}{2} v^{-\frac{1}{2}} f_X(\sqrt{v}) & : v > 4 \end{cases}$$

It is left to industrious students to verify that the two results agree.

7. A noisy communication channel.

A generic communication system transmits a known signal s over a channel that corrupts it with additive noise. This noise is often modeled as a Gaussian random variable, $X \sim \mathcal{N}(0, \sigma^2)$. Thus, the received signal is a random variable given by $R = s + X$.

- (a) What kind of random variable is R ? Find its CDF and pdf (write the CDF in terms of $\Phi(u)$, the CDF for a unit Gaussian random variable).

R is a continuous random variable with

$$f_R(r) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(r-s)^2}{2\sigma^2}}, \quad -\infty < r < \infty$$

Thus, R is Gaussian, $R \sim \mathcal{N}(s, \sigma^2)$.

- (b) Find the pdf of the random variable $Y = e^R$. Y is called a log-normal random variable, since $\ln Y$ is a normal (i.e., Gaussian) random variable. This type of random variable often appears in wireless communication problems.

With $Y = e^R$, it is clear that $Y > 0$. Thus, for $Y \leq 0$ we have $F_R(v) = f_R(v) = 0$. For $Y > 0$ we have

$$F_Y(v) = P\{Y \leq v\} = P\{e^R \leq v\} = P\{R \leq \ln v\} = F_R(\ln v) = \Phi\left(\frac{\ln v - s}{\sigma}\right).$$

To obtain the pdf, we simply differentiate and obtain for $v > 0$

$$f_Y(v) = \frac{d}{dv} \Phi\left(\frac{\ln v - s}{\sigma}\right) = \frac{d}{dv} \left(\frac{\ln v - s}{\sigma}\right) \times \phi\left(\frac{\ln v - s}{\sigma}\right) = \frac{1}{v\sigma} \phi\left(\frac{\ln v - s}{\sigma}\right) = \frac{1}{v\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln v - s)^2}{2\sigma^2}}.$$