

Continuous Random Variables
Expectation, Variance, and Uniform Random Variables

Assigned reading: *Ross* Chapter 5, Sections 1-3.
Noncredit Exercises: pp. 228-231: 1-4, 6-8, 10-14.

1. Properties of probability density functions.

For each of the following, indicate whether the specified function is a valid probability density function. Assume that the functions are zero outside of the specified ranges. For those functions that are not valid pdfs, state at least one property of pdfs that is not satisfied. Also, state whether there exists a constant C such that $Cf(x)$ is a valid pdf even though $f(x)$ is not.

- (a) $f(u) = |u|$ for $|u| < 1$. A valid pmf.
- (b) $f(u) = 1 - |u|$ for $|u| < 1$. A valid pmf.
- (c) $f(u) = \ln u$ for $0 < u < 1$. This is not a valid pmf, since the function is negative for $0 < u < 1$. However, $\int_0^1 \ln u \, du = u \ln u - u \Big|_0^1 = -1$, and therefore $-\ln u, u \in (0, 1)$ is a valid pmf.
- (d) $f(u) = \ln u$ for $0 < u < 2$. This is not a valid pmf, since the function is negative for $0 < u < 1$. Since the function is positive for $u \in (1, 2)$, $Cf(u)$ cannot be a valid pmf for any C .
- (e) $f(u) = 2u$ for $0 < u < 1$. A valid pmf.
- (f) $f(u) = \frac{2}{3}(u - 1)$ for $0 < u < 3$. Even though $\int_0^3 \frac{2}{3}(u - 1) \, du = \frac{1}{3}(u - 1)^2 \Big|_0^3 = 1$, this is not a valid pmf because the function is negative for $u \in (0, 1)$. And, since the function is positive for $u \in (1, 3)$, $Cf(u)$ cannot be a valid pmf for any C .
- (g) $f(u) = \exp(-2u)$ for $0 < u < \infty$. Since $\int_0^\infty \exp(-2u) \, du = \frac{1}{2}$, this is not a valid pmf. However, $2\exp(-2u), u \in (0, \infty)$ is a valid pmf (in fact, it is the exponential density with parameter 2).
- (h) $f(u) = 4\exp(-2u) - \exp(-u)$ for $0 < u < \infty$. Even though $\int_0^\infty 4\exp(-2u) - \exp(-u) \, du = 2 - 1 = 1$, this is not a valid pmf because the function is negative for $u > \ln 4$. Since the function is positive for $u \leq \ln 4$, $Cf(u)$ cannot be a valid pmf for any C .

2. Computing probabilities using density functions.

Let $\mathbf{X}$ denote a continuous random variable with pdf $f_{\mathbf{X}}(x)$ given by

$$f_{\mathbf{X}}(x) = \begin{cases} 1+x & : -1 < x \leq 0 \\ x & : 0 < x < 1 \\ 0 & : \text{else} \end{cases}$$

- (a) Find $P\{|\mathbf{X}| < \frac{1}{2}\}$. If one draws this density, it is immediately apparent that the answer is $\frac{1}{2}$. To find the same answer via integration,

$$\int_{-\frac{1}{2}}^0 1+u \, du + \int_0^{\frac{1}{2}} u \, du = u + \frac{u^2}{2} \Big|_{-\frac{1}{2}}^0 + \frac{u^2}{2} \Big|_0^{\frac{1}{2}} = \frac{1}{2} - \frac{1}{8} + \frac{1}{8} = \frac{1}{2}.$$

- (b) Find $P\{\mathbf{X} > 0 \mid |\mathbf{X}| < \frac{1}{2}\}$

$$P\{\mathbf{X} > 0 \mid |\mathbf{X}| < \frac{1}{2}\} = \frac{P\{\mathbf{X} > 0 \cap |\mathbf{X}| < \frac{1}{2}\}}{P\{|\mathbf{X}| < \frac{1}{2}\}} = \frac{P\{0 < \mathbf{X} < \frac{1}{2}\}}{P\{|\mathbf{X}| < \frac{1}{2}\}} = \frac{\frac{1}{2}}{\frac{1}{2}} = \frac{1}{4}, \text{ since } \int_0^{\frac{1}{2}} u \, du = \frac{1}{8}.$$

- (c) Find $E[\mathbf{X}]$.

$$E[\mathbf{X}] = \int_{-\infty}^{\infty} u f(u) \, du = \int_{-1}^0 u(1+u) \, du + \int_0^1 u(u) \, du = \frac{u^2}{2} + \frac{u^3}{3} \Big|_{-1}^0 + \frac{u^3}{3} \Big|_0^1 = \frac{1}{6}$$

- (d) Find $E[|\mathbf{X}|]$.

$$E[\mathbf{X}] = \int_{-\infty}^{\infty} u f(u) du = \int_{-1}^0 -u(1+u) du + \int_0^1 u(u) du = -\frac{u^2}{2} + -\frac{u^3}{3} \Big|_{-1}^0 + \frac{u^3}{3} \Big|_0^1 = \frac{1}{2}$$

3. Expectation of functions of random variables I.

Let $\mathbf{X}$ be a uniform random variable on the interval $[-1, 1]$.

- (a) For $\mathbf{Y} = \mathbf{X}^2$, compute the mean and variance of $\mathbf{Y}$.

$$E[\mathbf{Y}] = E[\mathbf{X}^2] = \int_{-1}^1 g(u) f(u) du = \int_{-1}^1 u^2 \frac{1}{2} du = \frac{1}{3}$$

$$var(\mathbf{Y}) = E[\mathbf{Y}^2] - E[\mathbf{Y}]^2 = \frac{1}{5} - \left(\frac{1}{3}\right)^2 = \frac{4}{45} \text{ since } E[\mathbf{Y}^2] = E[\mathbf{X}^4] = \int_{-1}^1 u^4 \frac{1}{2} du = \frac{1}{5}$$

- (b) Compute $E[\mathbf{Z}]$ for $\mathbf{Z} = g(\mathbf{X})$ with

$$g(x) = \begin{cases} x^2 & : x \geq 0 \\ -x^2 & : x < 0 \end{cases}$$

$$E[\mathbf{Z}] = \int_{-\infty}^{\infty} g(u) f(u) du = \int_{-1}^0 -\frac{1}{2} u^2 du + \int_0^1 \frac{1}{2} u^2 du = -\frac{1}{3} + \frac{1}{3} = 0$$

4. Expectation of functions of random variables II.

5. Consider a sphere whose radius is a random variable, R , with pdf $f_R(u) = 2u$ for $0 < u < 1$.

- (a) What is the average radius of the sphere?

Simply use the definition of expectation to obtain

$$E[R] = \int_0^1 2u^2 du = \frac{2}{3} u^3 \Big|_0^1 = \frac{2}{3}$$

- (b) What is the average volume of the sphere? If we call a sphere with average radius an average sphere, then does the average sphere have the average volume?

The volume of a sphere with radius R is given by $\text{Volume} = \frac{4}{3}\pi R^3$. Therefore, $E[\text{Volume}] = E[\frac{4}{3}\pi R^3] = \frac{4}{3}\pi E[R^3]$. Using proposition 2.1 we obtain

$$E[R^3] = \int_0^1 2u^4 du = \frac{2}{5} u^5 \Big|_0^1 = \frac{2}{5}$$

The volume of an average sphere is given by $\frac{4}{3}\pi E[R]^3 = \frac{4}{3}\pi (\frac{2}{3})^3 \neq \frac{4}{3}\pi \frac{2}{5} = E[\frac{4}{3}\pi R^3]$

- (c) What is the average surface area of the sphere? Does the average sphere have the average surface area?

The surface area of a sphere of radius R is given by $4\pi R^2$. Therefore, $E[\text{Surface Area}] = E[4\pi R^2] = 4\pi E[R^2]$. Using proposition 2.1 we obtain

$$E[R^2] = \int_0^1 2u^3 du = \frac{1}{2} u^4 \Big|_0^1 = \frac{1}{2}$$

The surface area of the average sphere is given by $4\pi E[R]^2 = 4\pi (\frac{2}{3})^2 \neq 4\pi \frac{1}{2} = 4\pi E[R^2]$

- (d) Show that $E[R] > E[R^2] > E[R^3]$ if the pdf for R is any valid pdf that is nonzero only on the unit interval $(0, 1)$.

Since $f(u) > 0$ only in the interval $(0, 1)$ we can use proposition 2.1 and write the expectation of any function of u as

$$E[g(u)] = \int_{-\infty}^{\infty} g(u) f(u) du = \int_0^1 g(u) f(u) du.$$

Now, for all $u \in (0, 1)$ we have $u > u^2 > u^3$. Therefore, for all $u \in (0, 1)$ we have $uf(u) \geq u^2 f(u) \geq u^3 f(u)$ where the equality holds only when $f(u) = 0$. Thus, we have

$$\int_0^1 u f(u) du \geq \int_0^1 u^2 f(u) du \geq \int_0^1 u^3 f(u) du,$$

and the equality holds only when $f(u) = 0$ for all $u \in (0, 1)$. But, since $f(u)$ is a pdf, it cannot be zero everywhere. Thus,

$$\int_0^1 u f(u) du = E[R] > \int_0^1 u^2 f(u) du = E[R^2] > \int_0^1 u^3 f(u) du = E[R^3].$$

6. The economics of supply and demand.

Let $\mathbf{X}$ be a random variable that denotes the weekly demand (measured in thousands of gallons) for gasoline at a particular gas station. The pdf for $\mathbf{X}$ is given by

$$f_{\mathbf{X}}(x) = \begin{cases} 5(1-x)^4 & : 0 < x < 1 \\ 0 & : \text{else} \end{cases}.$$

Let C denote the capacity (in thousands of gallons) of the station's storage tank, which is refilled weekly. Let $\mathbf{Y}$ be a random variable that denotes the amount (measured in gallons) of gasoline sold in a week. Note that the amount of gasoline sold cannot exceed the tanks capacity, regardless of the demand. Finally, assume that the gross profit for each gallon sold is \$0.64.

- (a) Suppose that $C = 0.5$ and for a particular week the random variable $\mathbf{X}$ takes the value 0.68. Can the gas station satisfy the demand?

No. The tank is empty after 500 gallons have been sold. So, the demand exceeds the available gas by 180 gallons.

- (b) Suppose that $C = 0.5$ and for a particular week the random variable $\mathbf{X}$ takes the value 0.43. Can the gas station satisfy the demand?

Yes. There are even 70 extra gallons.

- (c) Suppose that $C = 0.5$. What is the probability that the weekly demand for gasoline can be satisfied? Note: if your answer is, e.g., 0.25, then, in the long run, the gas station can supply demand only one out of four weeks.

The weekly demand is satisfied if it does not exceed 0.5. Thus, $P\{\mathbf{X} \leq 0.5\} = \int_0^{0.5} 5(1-u)^4 du = -(1-u)^5 \Big|_{-1}^0 = 1 - \left(\frac{1}{2}\right)^5 = \frac{31}{32}$.

- (d) What is the minimum value for C that will ensure the probability that the demand exceeds the supply is no larger than 10^{-5} ?

We want the smallest C that gives $P\{\mathbf{X} > C\} \leq 10^{-5}$. So, $P\{\mathbf{X} > C\} = \int_C^1 5(1-u)^4 du = -(1-u)^5 \Big|_C^1 = (1-C)^5$. Now we have $(1-C)^5 \leq 10^{-5}$ if $(1-C) \leq 10^{-1} \rightarrow C \geq 0.9$. Thus, a 900 gallon tank is required.

- (e) How is $\mathbf{Y}$ related to $\mathbf{X}$?

$$\mathbf{Y} = \begin{cases} \mathbf{X} & : \mathbf{X} \leq C \\ C & : \text{else} \end{cases}$$

- (f) What is the expected value of weekly gross profit?

The weekly gross profit is $640\mathbf{Y}$, so the expected value of the weekly gross profit is given by

$$E[640\mathbf{Y}] = 640E[\mathbf{Y}] \text{ with } E[\mathbf{Y}] = \int_0^C u \cdot 5(1-u)^4 du + \int_C^1 C \cdot 5(1-u)^4 du$$

The first integral can be solved using integration by parts: $\int w dv = wv - \int v dw$ with $w = u$ (so $dw = du$) and $v = -(1-u)^5$ so that $dv = 5(1-u)^4 du$. This gives

$$\int u \cdot 5(1-u)^4 du = -u(1-u)^5 - \int -(1-u)^5 du = -u(1-u)^5 - \frac{1}{6}(1-u)^6$$

and from this we obtain

$$\int_0^C u \cdot 5(1-u)^4 du = \left[-u(1-u)^5 - \frac{1}{6}(1-u)^6 \right] \Big|_0^C = -C(1-C)^5 - \frac{1}{6}(1-C)^6 + \frac{1}{6}$$

The second integral is more easily solved $\int_C^1 C \cdot 5(1-u)^4 du = -C(1-u)^5 \Big|_C^1 = C(1-C)^5$.

Putting these results together we obtain: $E[\mathbf{Y}] = [-C(1-C)^5 - \frac{1}{6}(1-C)^6 + \frac{1}{6}] + C(1-C)^5 = -\frac{1}{6}(1-C)^6 + \frac{1}{6}$. Thus, the expected value of the gross profit is $E[640\mathbf{Y}] = \frac{640}{6} [1 - (1-C)^6]$

- (g) Suppose that the owner pays \$ $20C$ as a weekly rent on a tank of capacity $1000C$ gallons (with $0 \leq C \leq 1$). What is the expected value of the weekly net profit? What value of C will maximize this?

Net profit is the gross profit less expenses, so the expected value of the gross profit is

$$E[640\mathbf{Y} - 20C] = E[640\mathbf{Y}] - 20C = \frac{640}{6} [1 - (1 - C)^6] - 20C.$$

To maximize this, simply differentiate with respect to C and set the result to zero:

$$\begin{aligned} 0 &= \frac{d}{dC} \left[\frac{640}{6} [1 - (1 - C)^6] - 20C \right] \\ &= 640(1 - C)^5 - 20 \\ \frac{1}{32} &= (1 - C)^5 \\ \frac{1}{2} &= (1 - C) \\ \frac{1}{2} &= C \end{aligned}$$