

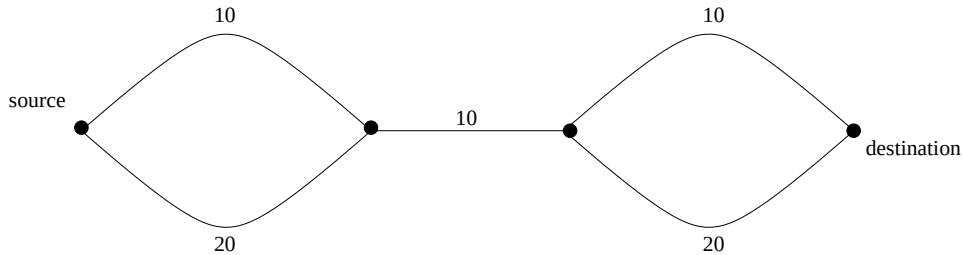
Discrete Random Variables: Expectation, Variance, and Binomial Random Variables

Assigned reading: Ross Chapter 4, Sections 1-6.

Noncredit Exercises: pp.171-180: 3-5, 7, 8, 12, 13, 14, 20, 24, 35, 40, 41, 43, 44.

1. Network Capacity and Failure

Consider the following network, with link capacities (in packets) as shown.



Assume that each link fails independently with probability $p = 1/2$.

- (a) What is the probability that a message can be sent successfully from the *source* to the *destination*?

The probability of success is the probability that one of the first two parallel links will function AND the middle link will function AND one of the last two parallel links will function:

$$(p + p - p^2)(p)(p + p - p^2) = \frac{3}{4} \frac{1}{2} \frac{3}{4} = \frac{9}{32}.$$

- (b) Let X denote the number of packets that can be sent from the *source* to the *destination*. Determine the pmf for X .

If a message can be sent, then 10 messages can be sent (any path through the network has capacity at least 10). More than 10 messages can never be sent, since the capacity of the middle link is only 10. Thus, X has two possible values, and using the results of part (a) we have

$$p_X(10) = P\{\text{success}\} = \frac{9}{32}, \quad p_X(0) = P\{\text{failure}\} = 1 - \frac{9}{32} = \frac{23}{32}$$

- (c) Determine the expected value of X .

Simply use the definition of expectation

$$E[X] = 0 \times p_X(0) + 10 \times p_X(10) = \frac{90}{32} = \frac{45}{16}$$

2. Expectation and Variance

Let the random variable X be the number of passengers that show up for a five person flight. Suppose that X takes on the integer values $\{0, 1, \dots, 8\}$, and has pmf given by $p_X(0) = p_X(1) = p_X(7) = p_X(8) = 0.05$, $p_X(2) = p_X(3) = p_X(5) = p_X(6) = 0.15$, $p_X(4) = 0.2$.

Let Z denote the number of passengers who are left behind (i.e., $Z = 0$ if $X \leq 5$ and $Z = X - 5$ if $X > 5$).

- (a) Compute $E[Z]$, the expected number of people left behind.

This is a simple application of Proposition 4.1.

$$\begin{aligned}
 E[Z] &= \sum_{x=0}^8 Z(x)p_X(x) \\
 &= Z(0) \times .05 + Z(1) \times .05 + Z(2) \times .15 + Z(3) \times .15 + Z(4) \times .2 + Z(5) \times .15 \\
 &\quad + Z(6) \times .15 + Z(7) \times .05 + Z(8) \times .05 \\
 &= 0 \times .05 + 0 \times .05 + 0 \times .15 + 0 \times .15 + 0 \times .2 + 0 \times .15 + 1 \times .15 + 2 \times .05 + 3 \times .05 = 0.4
 \end{aligned}$$

- (b) Compute $Var(Z)$.

To solve this, use the equation $Var(Z) = E[Z^2] - E[Z]^2$.

$$E[Z^2] = \sum_{x=0}^8 Z(x)^2 p_X(x) = 1 \times .15 + 4 \times .05 + 9 \times .05 = 0.8$$

Therefore, $Var(Z) = E[Z^2] - E[Z]^2 = 0.8 - 0.16 = 0.64$.

The airline has decided to run a two-for-one special: Each ticketed passenger can bring along one friend at no charge. This special is so attractive that every ticketed passenger who shows up brings along one friend. Assume that a ticketed passenger (and also the friend) will decline to fly if there is not room for both on the plane. Let X be the number of ticketed passengers who show up, with pmf as above, and let W be the random variable that denotes the number of people left behind (i.e., the total of ticketed passengers and their friends who show up and are left behind).

- (c) What is $E[W]$, the expected number of people who are left behind?

This is another fairly simple application of Proposition 4.1:

$$\begin{aligned}
 E[W] &= \sum_{x=0}^8 W(x)p_X(x) \\
 &= W(0) \times .05 + W(1) \times .05 + W(2) \times .15 + W(3) \times .15 + W(4) \times .2 + W(5) \times .15 \\
 &\quad + W(6) \times .15 + W(7) \times .05 + W(8) \times .05 \\
 &= 0 \times .05 + 0 \times .05 + 0 \times .15 + 2 \times .15 + 4 \times .2 + 6 \times .15 + 8 \times .15 + 10 \times .05 + 12 \times .05 = 4.3
 \end{aligned}$$

- (d) What is $Var(W)$?

Again, we will use $Var(W) = E[W^2] - E[W]^2$

$$\begin{aligned}
 E[W^2] &= \sum_{x=0}^8 W(x)^2 p_X(x) \\
 &= 4 \times .15 + 9 \times .2 + 25 \times .15 + 49 \times .15 + 91 \times .05 + 121 \times .05 = 31
 \end{aligned}$$

Therefore, $Var(W) = E[W^2] - E[W]^2 = 31 - (4.3)^2 = 12.51$.

3. The Binomial Random Variable I

Problem 40, pg. 176 of Ross.

$$\binom{5}{4} \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^1 + \left(\frac{1}{3}\right)^5 = \frac{11}{243}$$

4. The Binomial Random Variable II

Problem 42, pg. 176 of Ross.

$$\begin{aligned}
 \binom{5}{3} p^3(1-p)^2 + \binom{5}{4} p^4(1-p)^1 + \binom{5}{5} p^5(1-p)^0 &\geq \binom{3}{2} p^2(1-p)^1 \binom{3}{3} p^3(1-p)^0 \\
 10p^3(1-p)^2 + 5p^4(1-p) + p^5 &\geq 3p^2(1-p) + p^3 \\
 6p^3 - 15p^2 + 12p - 3 &\geq 0 \\
 6(p - \frac{1}{2})(p - 1)^2 &\geq 0 \\
 p &\geq \frac{1}{2}
 \end{aligned}$$

5. The Binomial Random Variable III

GoGoDotCom Inc., is an investment management service that has the stated goal of doubling the value of their clients' investments in a week via day trading. Their TV commercial boasts that, "On average, our clients triple their money in five weeks!" After poring over back issues of the *Wall Street Journal* you learn the truth: at the end of any week, GoGoDotCom will have doubled your investment with probability 0.5, and it will have halved your investment with probability 0.5. Thus, at the end of the first week, an initial investment of \$32 will be worth either \$64 or \$16, each with probability 0.5. Performance in any week is independent of performance during the other weeks. Anxious to apply your new skills in probability theory, you decide to invest \$32, and to let that investment ride for five weeks (in fact, you decide not to even look at the stock prices until the five weeks are over). Let the random variable X denote the value in dollars of your investment at the end of a five week period.

- (a) What are the possible values of X ?

The investment doubles on Y weeks, and is halved on the remaining $Y - 5$ weeks. Thus, the total return can be written as $X = 32 \times 2^Y \times (1/2)^{Y-5} = 32 \times 2^{2Y-5}$. Thus, for values of Y in $\{0, 1, 2, 3, 4, 5\}$, X takes on values in $\{1, 4, 16, 64, 256, 1024\}$.

- (b) What is the pmf of the random variable X ?

The pmf for Y is given by:

$$p_Y(k) = \binom{5}{k} (1/2)^k (1/2)^{5-k} = \binom{5}{k} \times 1/32$$

Therefore, we have $p_X(1) = p_Y(0) = 1/32$, $p_X(4) = p_Y(1) = 5/32$, $p_X(16) = p_Y(2) = 10/32$, $p_X(64) = p_Y(3) = 10/32$, $p_X(256) = p_Y(4) = 5/32$, $p_X(1024) = p_Y(5) = 1/32$.

- (c) What is the expected value of X ? Is the TV commercial accurate?

$$\begin{aligned} E[X] &= \sum_x x p_X(x) \\ &= 1 \times 1/32 + 4 \times 5/32 + 16 \times 10/32 + 64 \times 10/32 + 256 \times 5/32 + 1024 \times 1/32 = 3125/32 = 97.65625 \end{aligned}$$

This is more than $3 \times \$32$, so the commercial actually understates the true performance.

- (d) What is the probability that you will lose money on your investment?

$$P\{X < 32\} = p_X(1) + p_X(4) + p_X(16) = 16/32 = 1/2.$$