

1. Sensor fusion

(a) The likelihood matrix for observation XY given the hypothesis is the following.

$XY \rightarrow$	00	01	02	10	11	12	20	21	22
H_0	<u>0.56</u>	<u>0.16</u>	0.08	<u>0.07</u>	0.02	0.01	<u>0.07</u>	0.02	0.01
H_1	0.01	0.01	<u>0.08</u>	0.03	<u>0.03</u>	<u>0.24</u>	0.06	<u>0.06</u>	<u>0.48</u>

The ML decisions are indicated by the underlined elements in the likelihood matrix. The larger number in each column is underlined, with the tie broken in case 02 broken in favor of H_1 , as specified in the problem statement.

(b) For the ML rule, p_{false_alarm} is the sum of the entries in the row for H_0 in the likelihood matrix that are not underlined. So $p_{false_alarm} = 0.08 + 0.02 + 0.01 + 0.02 + 0.01 = 0.14$.

For the ML rule, p_{miss} is the sum of the entries in the row for H_1 in the likelihood matrix that are not underlined. So $p_{miss} = 0.01 + 0.01 + 0.03 + 0.06 = 0.11$.

(c) The joint probability matrix is given by

$XY \rightarrow$	00	01	02	10	11	12	20	21	22
H_0	<u>0.448</u>	<u>0.128</u>	<u>0.064</u>	<u>0.056</u>	<u>0.016</u>	0.008	<u>0.056</u>	<u>0.016</u>	0.008
H_1	0.002	0.002	0.016	0.006	0.006	<u>0.048</u>	0.012	0.012	<u>0.096</u>

The MAP decisions are indicated by the underlined elements in the joint probability matrix. The larger number in each column is underlined.

(d) For the MAP rule,

$$p_{false_alarm} = P[XY \in \{12, 22\} | H_0] = 0.01 + 0.01 = 0.02,$$

and

$$p_{miss} = P[XY \notin \{12, 22\} | H_1] = 1 - P[XY \in \{12, 22\} | H_1] = 1 - 0.24 - 0.48 = 0.28.$$

Thus, for the MAP rule, $p_e = (0.8)(0.02) + (0.2)(0.28) = 0.072$. (Check that for the MAP rule, p_e is the sum of the probabilities in the joint probability matrix that are not underlined.)

(e) Using the conditional probabilities found in (a) and the given values of π_0 and π_1 yields that for the ML rule: $p_e = (0.8)(0.14) + (0.2)(0.11) = 0.134$ (Note: While the MAP rule has a smaller probability of error than the ML rule, the p_{miss} for the MAP rule might be unacceptably high. For this problem, it may be appropriate to take the costs of wrong decisions into account.)

2. Simple discrete random variables

(a)

	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
$i = 1$	1	2	3	4	5	6
$i = 2$	2	2	3	4	5	6
$i = 3$	3	3	3	4	5	6
$i = 4$	4	4	4	4	5	6
$i = 5$	5	5	5	5	5	6
$i = 6$	6	6	6	6	6	6

(b) The range of Y is $\{1, 2, 3, 4, 5, 6\}$.

(c) By inspection of the table, we see that $|Y = 1| = 1$, $|Y = 2| = 3$, $|Y = 3| = 5$, $|Y = 4| = 7$, $|Y = 5| = 9$, and $|Y = 6| = 11$, which we can compactly write as $|Y = i| = 2i - 1$. Thus, the probability mass function of Y is given by $p(i) = \frac{2i-1}{36}$ for $i \in \{1, 2, 3, 4, 5, 6\}$, and $p(i) = 0$ otherwise.

3. Computing a simple probability mass function

Define the events for $i = 0$ or $i = 1$:

$$\begin{aligned} D_i &= \text{"Salesman sold deluxe model on } i^{th} \text{ appointment"} \\ S_i &= \text{"Salesman sold standard model on } i^{th} \text{ appointment"} \\ N_i &= \text{"Salesman sold nothing on } i^{th} \text{ appointment"} \end{aligned}$$

Then

$$\begin{aligned}
p(0) &= P(N_1 N_2) = (1 - 0.3)(1 - 0.6) = 0.28 \\
p(500) &= P((S_1 N_2) \cup (N_1 S_2)) = \left(\frac{0.3}{2}\right)(1 - 0.6) + (1 - 0.3)\left(\frac{0.6}{2}\right) = 0.27 \\
p(1000) &= P((D_1 N_2) \cup (S_1 S_2) \cup (N_1 D_2)) = \left(\frac{0.3}{2}\right)(1 - 0.6) + \left(\frac{0.3}{2}\right)\left(\frac{0.6}{2}\right) + (1 - 0.3)\left(\frac{0.6}{2}\right) = 0.315 \\
p(1500) &= P((D_1 S_2) \cup (S_1 D_2)) = 2\left(\frac{0.3}{2}\right)\left(\frac{0.6}{2}\right) = 0.09 \\
p(2000) &= P(D_1 D_2) = \left(\frac{0.3}{2}\right)\left(\frac{0.6}{2}\right) = 0.045
\end{aligned}$$

4. Maximum likelihood parameter estimation

(a)

$$\begin{aligned}
P((2, 1, 0, 2, 0, 1, 2, 2, 0, 2)) &= \left(\frac{1+a}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1-a}{3}\right)\left(\frac{1+a}{3}\right)\left(\frac{1-a}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1+a}{3}\right)\left(\frac{1+a}{3}\right)\left(\frac{1-a}{3}\right)\left(\frac{1+a}{3}\right) \\
&= \left(\frac{1-a}{3}\right)^3 * \left(\frac{1}{3}\right)^2 \left(\frac{1+a}{3}\right)^5.
\end{aligned}$$

(b) Finding a to maximize $\left(\frac{1-a}{3}\right)^3 * \left(\frac{1}{3}\right)^2 \left(\frac{1+a}{3}\right)^5$ yields $\hat{a}_{MLE} = 0.2$ (See part (c) for general case.)

(c) Same answer as (b). The order of the output values does not effect the probability of the total output vector.

(d) In general, the probability of a particular output sequence that has n_0 0's, n_1 1's, and n_2 2's, satisfies

$$\left(\frac{1-a}{3}\right)^{n_0} + \left(\frac{1}{3}\right)^{n_1} + \left(\frac{1+a}{3}\right)^{n_2} = \frac{(1-a)^{n_1}(1+a)^{n_2}}{3^n}$$

. Maximizing this with respect to a over the interval $[-1, 1]$ is equivalent to maximizing $(1-a)^{n_0}(1+a)^{n_2}$. Or, taking the log, the problem is to find a to maximize $n_0 \log(1-a) + n_2 \log(1+a)$. The derivative of the function is

$$\frac{d(n_0 \log(1-a) + n_2 \log(1+a))}{da} = \frac{-n_0}{1-a} + \frac{n_2}{1+a}.$$

In the trivial special case that $n_0 = n_2 = 0$, the probability of the observation is the same for all a , in which case $\hat{a}_{ML}$ can be chosen arbitrarily from the interval $[-1, 1]$. (We won't take off credit if this fine point is missed, but congratulations if you spotted it!) Otherwise the derivative is strictly decreasing in a and is zero only at $a = \frac{n_2 - n_0}{n_2 + n_0}$. The maximum of the function thus occurs at that value, so $\hat{a}_{ML} = \frac{n_2 - n_0}{n_2 + n_0}$.

5. Bayes decision making for given cost

(a)

$$\Lambda(n) = \begin{cases} 25 & \text{if } 1 \leq n \leq 300 \\ 1.25 & \text{if } 301 \leq n \leq 400 \\ 0 & \text{if } 401 \leq n \leq 780 \end{cases}$$

(b) The ML rule is given by the LRT (see previous problem set) with the threshold one. So the LRT decides H_1 is true if $1 \leq X \leq 400$ and decides H_0 is true otherwise. For this rule,

$$\begin{aligned}
p_{false_alarm} &= P[1 \leq X \leq 400 | H_0] = (300)(0.0001) + (100)(0.002) = 0.23 \\
p_{miss} &= P[401 \leq X \leq 780 | H_1] = 0
\end{aligned}$$

(c) The MAP rule is given by the LRT (see previous problem set) with the threshold $\frac{\pi_0}{\pi_1} = 9,999$. Since this is always larger than the likelihood ratio, the MAP rule always decides that H_0 is true. For this rule, $p_{false_alarm} = 0$ and $p_{miss} = 1$. Therefore, $p_e = \pi_0 p_{false_alarm} + \pi_1 p_{miss} = 10^{-4}$. Note: while the unconditional error probability is small, the sensor is useless!

(d) As mentioned in the hint, the Bayes minimum cost rule is given by the LRT with threshold τ given by

$$\tau = \frac{\pi_0(C_{10} - C_{00})}{\pi_1(C_{01} - C_{11})} = \frac{(1-10)^{-4}(1-0)}{10^{-4}(1000-0)} = 9.999$$

So the Bayes minimum cost rule decides H_1 is true if $1 \leq X \leq 300$ and decides H_0 is true otherwise. For this rule,

$$\begin{aligned} p_{false_alarm} &= P[1 \leq X \leq 300|H_0] = (300)(0.0001) = 0.03 \\ p_{miss} &= P[301 \leq X \leq 780|H_1] = P[301 \leq X \leq 400|H_1] = 0.25 \end{aligned}$$

Since $C_{00} = C_{11} = 0$, the resulting average cost is

$$\begin{aligned} C_{ave} &= C_{10}P[\{\text{decide } H_1 \text{ is true}\} \cap H_0] + C_{01}P[\{\text{decide } H_0 \text{ is true}\} \cap H_1] \\ &= C_{10}\pi_0 p_{false_alarm} + C_{01}\pi_1 p_{miss} \\ &= (1)(1 - 10^{-4})(0.03) + (1000)(10^{-4})(0.25) \approx 0.055 \end{aligned}$$