

1. Independent vs. mutually exclusive

(a) For each pair of events we check whether the probability of the intersection of the two events is equal to the probability of the product. For each pair of events the probability of the intersection is listed in the upper right part of the following table.

	$P[B] = \frac{4}{9}$	$P[C] = \frac{1}{6}$	$P[D] = \frac{5}{18}$
$P[A] = \frac{1}{6}$	$\frac{1}{18}$	0	$\frac{1}{18}$
$P[B] = \frac{4}{9}$	—	$\frac{1}{9}$	$\frac{1}{6}$
$P[C] = \frac{1}{6}$	—	—	0

Examination yields that NO two of these events are independent.

(b) Since all 36 outcomes in S have positive probability, a set is empty if and only if it has probability zero. Thus 0's in the table indicate an empty intersection. So A & C and C & D are the only choices of two events such that the two events are mutually exclusive. (Note that no three of three of the events are mutually exclusive.)

2. Sequence of events coming true

Let A_i denote the event that the student passes the i -th exam. Then we are given the following (conditional) probabilities:

$$P(A_1) = 0.9 \quad P(A_2|A_1) = 0.8 \quad P(A_3|A_1A_2) = 0.7$$

Based on this information, we can answer the questions in this problem as follows.

1. Let $\mathcal{A}$ be the event that the student passes all three exams. Then, using the chain rule for conditional probabilities, we obtain

$$P(\mathcal{A}) = P(A_1A_2A_3) = P(A_3|A_1A_2)P(A_2|A_1)P(A_1) = (0.7)(0.8)(0.9) = 0.504$$

2. Now let B_i denote the event that the student fails the i -th exam. Notice that $B_1 = A_1^c$, but $B_2 \neq A_2^c$. This is so because the events A_2 and B_2 can occur only if the event A_1 occurs, namely if the student passes the first exam. Thus the events A_2 and B_2 form a partition of the event A_2 , not of the sample space Ω .

Here, we are asked for the conditional probability of the event B_2 , given the event $\mathcal{A}^c$. Notice that if B_2 occurs, namely the student fails the second exam, then she cannot pass all three exams and $\mathcal{A}^c$ definitely occurs. Thus $P(\mathcal{A}^c|B_2) = 1$. Using Bayes inversion formula, we obtain:

$$P(B_2|\mathcal{A}^c) = \frac{P(\mathcal{A}^c|B_2)P(B_2)}{P(\mathcal{A}^c)} = \frac{P(B_2)}{1 - P(\mathcal{A})} = \frac{P(B_2)}{0.496} \quad (1)$$

Now let us compute the probability $P(B_2)$ that the student fails the second exam, by conditioning on the event A_1 . Notice that if the event $A_1^c = B_1$ occurs, then the student does not take the second exam, and neither of the events A_2 or B_2 occurs. Thus $P(B_2|A_1^c) = 0$. On the other hand, since A_2 and B_2 partition the event A_1 , we can conclude that $P(B_2|A_1) = 1 - P(A_2|A_1) = 0.2$. We therefore obtain

$$P(B_2) = P(B_2|A_1)P(A_1) + P(B_2|A_1^c)P(A_1^c) = (0.2)(0.9) + (0)(0.1) = 0.18$$

using the theorem of total probability. This is all we need. Substituting this result into (1), we find that $P(B_2|\mathcal{A}^c) = (0.18)/(0.496) = 45/124 \simeq 0.3629$.

3. A posteriori probability of a cause

Let E denote the event that a randomly chosen pregnant woman has an ectopic pregnancy, and let S be the event that she is a smoker. Then the information given to us in the problem states that $P(E|S) = 2P(E|S^c)$ and $P(S) = 0.32$. Using Bayes inversion formula and the theorem of total probability, we conclude that

$$\begin{aligned} P(S|E) &= \frac{P(E|S)P(S)}{P(E|S)P(S) + P(E|S^c)P(S^c)} = \frac{2P(E|S^c)P(S)}{2P(E|S^c)P(S) + P(E|S^c)P(S^c)} \\ &= \frac{2P(S)}{2P(S) + P(S^c)} = \frac{0.64}{0.64 + (1 - 0.32)} = \frac{16}{33} \simeq 0.4848 \end{aligned}$$

4. Who voted?

Let $\mathcal{I}, \mathcal{L}, \mathcal{C}$ denote the events that a randomly chosen voter in our city is an Independent, a Liberal, and a Conservative, respectively. Then it is given that

$$P(\mathcal{I}) = 0.46 \quad P(\mathcal{L}) = 0.30 \quad P(\mathcal{C}) = 0.24$$

Since the sample space is finite, the events $\mathcal{I}, \mathcal{L}, \mathcal{C}$ are disjoint, and $P(\mathcal{I}) + P(\mathcal{L}) + P(\mathcal{C}) = 1$, we can conclude that these events form a partition of the sample space. Now let $\mathcal{V}$ be the event that a randomly chosen voter in our city voted in the recent election. Then we are also given the conditional probabilities:

$$P(\mathcal{V}|\mathcal{I}) = 0.35 \quad P(\mathcal{V}|\mathcal{L}) = 0.62 \quad P(\mathcal{V}|\mathcal{C}) = 0.58$$

Based on this information, all the questions in this problem can be answered straightforwardly, using Bayes inversion formula and the theorem of total probability.

1.

$$\begin{aligned} P(\mathcal{V}) &= P(\mathcal{V}|\mathcal{I})P(\mathcal{I}) + P(\mathcal{V}|\mathcal{L})P(\mathcal{L}) + P(\mathcal{V}|\mathcal{C})P(\mathcal{C}) \\ &= (0.35)(0.46) + (0.62)(0.30) + (0.58)(0.24) = 0.4862 \end{aligned}$$

2.

$$P(\mathcal{I}|\mathcal{V}) = \frac{P(\mathcal{V}|\mathcal{I})P(\mathcal{I})}{P(\mathcal{V})} = \frac{(0.35)(0.46)}{0.4862} \simeq 0.331$$

3.

$$P(\mathcal{L}|\mathcal{V}) = \frac{P(\mathcal{V}|\mathcal{L})P(\mathcal{L})}{P(\mathcal{V})} = \frac{(0.62)(0.30)}{0.4862} \simeq 0.383$$

4.

$$P(\mathcal{C}|\mathcal{V}) = \frac{P(\mathcal{V}|\mathcal{C})P(\mathcal{C})}{P(\mathcal{V})} = \frac{(0.58)(0.24)}{0.4862} \simeq 0.286$$

5. Independent vs. mutually exclusive, II

1. If E is an event independent of itself, then $P(E) = P(E \cap E) = P(E)P(E) = [P(E)]^2$. This can happen if $P(E) = 0$. If $P(E) \neq 0$ then cancelling a factor of $P(E)$ on each side yields $P(E) = 1$. In summary, either $P(E) = 0$ or $P(E) = 1$.

2. In general, we have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. If the events A and B are independent, then

$$P(A \cup B) = P(A) + P(B) - P(A)P(B) = 0.3 + 0.4 - (0.3)(0.4) = 0.58$$

On the other hand, if the events A and B are mutually exclusive, then $P(A \cap B) = 0$ and therefore $P(A \cup B) = 0.3 + 0.4 = 0.7$.

3. If $P(A) = 0.6$ and $P(B) = 0.8$, then the two events could be independent. However, they could not be mutually exclusive since then $P(A \cap B)$ would be zero, and you would have to conclude that $P(A \cup B) = P(A) + P(B) = 1.4$, which is nonsense.

6. Bayes' formula – your turn

(a) Given mutually exclusive events $F_1, F_2, \dots, F_n$ such that $S = F_1 \cup F_2 \cup \dots \cup F_n$ (here S is the space of all outcomes) and if E is an event such that $P[E] > 0$, then

$$P[F_i|E] = \frac{P[E|F_i]P[F_i]}{\sum_j P[E|F_j]P[F_j]}$$

(One might also note that $P[F_i|E] = \frac{P[E|F_i]}{P[E]}$ as a useful intermediate step.)

(b) See hint.

7. Single sensor

(a) The ML decisions are indicated by the underlined elements in the likelihood matrix.

$Z \rightarrow$	0	1	2	3	4
H_0	<u>0.80</u>	<u>0.08</u>	0.06	0.04	0.02
H_1	0.01	0.02	<u>0.07</u>	<u>0.10</u>	<u>0.80</u>

The larger number in each column is underlined. The likelihood ratio for the possible observation values is given by $(\Lambda(0), \Lambda(1), \Lambda(2), \Lambda(3), \Lambda(4)) = (0.0125, 0.25, 1.167, 2.5, 40)$. For example, if it is observed that $Z = 3$ then the likelihood ratio is 3. We see that $\Lambda(i) > 1$ if and only if $i \in \{2, 3, 4\}$, so that the ML rule indicated in the table is equivalent to the LRT with threshold $\tau = 1$.

(b) For the ML rule, p_{false_alarm} is the sum of the entries in the row for H_0 in the likelihood matrix that are not underlined. So $p_{false_alarm} = 0.06 + 0.04 + 0.02 = 0.12$.

For the ML rule, p_{miss} is the sum of the entries in the row for H_1 in the likelihood matrix that are not underlined. So $p_{miss} = 0.01 + 0.02 = 0.03$.

(c) The joint probability matrix is given by

$Z \rightarrow$	0	1	2	3	4
H_0	<u>0.640</u>	<u>0.064</u>	<u>0.048</u>	<u>0.032</u>	0.016
H_1	0.002	0.004	0.014	0.020	<u>0.16</u>

The MAP decisions are indicated by the underlined elements in the joint probability matrix. The larger number in each column is underlined.

We see that $\frac{\pi_0}{\pi_1} = 4$ and $\Lambda(i) > 4$ if and only if $i = 4$, so that the MAP rule indicated in the table is equivalent to the LRT with threshold $\tau = 4$.

(d) For the MAP rule,

$$p_{false_alarm} = P[Z = 4|H_0] = 0.02$$

and

$$p_{miss} = P[Z \in \{0, 1, 2, 3\}|H_1] = 1 - P[Z = 4|H_1] = 1 - 0.80 = 0.20.$$

Thus, also for the MAP rule, $p_{ave} = (0.8)(0.02) + (0.2)(0.20) = 0.056$. (One also has that p_{ave} is the sum of the probabilities in the joint probability matrix that are not underlined.)

(e) Using the conditional probabilities found in (a) and the given values of π_0 and π_1 yields that for the ML rule: $p_{ave} = (0.8)(0.12) + (0.2)(0.03) = 0.102$ (Note: While the MAP rule has a smaller average probability of error than the ML rule, the p_{miss} for the MAP rule might be unacceptably high. For this problem, it may be appropriate to take the costs of wrong decisions into account. See next problem set.)