

1. Randomly selected die

Let E_i be the event that the die with i red faces is drawn from the bag, for $i = 1, 2$, or 3 . Let R_j denote the event that red comes up on the j th role of the die.

(a) $P[R_1] = P[R_1|E_1]P[E_1] + P[R_1|E_2]P[E_2] + P[R_1|E_3]P[E_3] = \frac{1}{6}\frac{1}{3} + \frac{2}{6}\frac{1}{3} + \frac{3}{6}\frac{1}{3} = \frac{1}{3}$

(b) We need to find $P[R_2|R_1]$.

First, $P[R_1R_2] = P[R_1R_2|E_1]P[E_1] + P[R_1R_2|E_2]P[E_2] + P[R_1R_2|E_3]P[E_3] = (\frac{1}{6})^2\frac{1}{3} + (\frac{2}{6})^2\frac{1}{3} + (\frac{3}{6})^2\frac{1}{3} = \frac{7}{54}$.

Therefore, $P[R_2|R_1] = \frac{P[R_1R_2]}{P[R_1]} = \frac{7}{18}$. (Note: we have essentially used Bayes' formula.)

(c) We need to find $P[E_3|R_1R_2R_3]$.

Now $P[E_3R_1R_2R_3] = P[E_3]P[R_1R_2R_3|E_3] = \frac{1}{3}(\frac{3}{6})^3 = \frac{1}{24}$, and

$P[R_1R_2R_3] = P[R_1R_2R_3|E_1]P[E_1] + P[R_1R_2R_3|E_2]P[E_2] + P[R_1R_2R_3|E_3]P[E_3] = (\frac{1}{6})^3\frac{1}{3} + (\frac{2}{6})^3\frac{1}{3} + (\frac{3}{6})^3\frac{1}{3} = \frac{1}{18}$.

Therefore, $P[E_3|R_1R_2R_3] = \frac{P[E_3R_1R_2R_3]}{P[R_1R_2R_3]} = \frac{18}{24} = \frac{3}{4}$. Note : we have essentially used Bayes formula.

2. Ternary channel

(a)

$$P[s_j \text{ received}] = \sum_{i=1}^3 P[s_j \text{ received} | s_i \text{ sent}] \cdot P[s_i \text{ sent}].$$

Since $P[s_i \text{ sent}] = 1/3$ for all i , the required probabilities above are found to be

	s_1	s_2	s_3
$P[s_j \text{ received}]$	0.29	0.36	0.35

Now, we compute $P[s_i \text{ sent} | s_j \text{ received}]$. For example,

$$P[s_1 \text{ sent} | s_1 \text{ received}] = \frac{P[s_1 \text{ received} | s_1 \text{ sent}] \cdot P[s_1 \text{ sent}]}{P[s_1 \text{ received}]} = \frac{0.8 \cdot 1/3}{0.29} = 0.9195.$$

In this fashion, we can form a table similar to the one given in the problem, but consisting of the inverse conditional probabilities $P[s_i \text{ sent} | s_j \text{ received}]$:

		Send, i		
		s_1	s_2	s_3
Receive, j	s_1	0.920	0.093	0.095
	s_2	0.057	0.8333	0.048
	s_3	0.023	0.074	0.857

3. Matching problem

(a) Let E be the event that the router gets at least 6 packets routed correctly, and E_i be the event that the router gets exactly i packets routed correctly. We see that $E = E_6 \cup E_8$ (what about E_7 ?). Event E_8 contains only one of the $8!$ possible permutations, and event E_6 contains $\binom{8}{6} = 28$ permutations, each corresponding to a choice of 2 out of 8 packets to send to wrong stations by swapping the two packets. Therefore, $P(E) = \frac{1+28}{8!} = \frac{29}{40320}$.

(b) We “renumber” the stations $A, B, \dots, H$ as $1, 2, \dots, 8$, and define the following events:

$$\begin{aligned} A &= \{\text{Station 1 gets the wrong packet}\} \\ B &= \{\text{No station gets its desired packet}\} \\ B_i &= \{\text{Station } i \text{ gets the right packet}\} \end{aligned}$$

It is clear that the probability we seek is $P(B | A) = P(BA)/P(A) = P(B)/P(A)$, since $B \subset A$. It is easier to find the probability of the complementary event $P(B^c)$, the probability that at least one station gets its

designated packet, than $P(B)$. Now $P(B^c) = P(B_1 \cup B_2 \cup \dots \cup B_8)$, and from the formula for the probability of the union of events we know that

$$P(B_1 \cup B_2 \cup \dots \cup B_8) = \sum_i P(B_i) - \sum_{i_1 < i_2} P(B_{i_1} B_{i_2}) + \sum_{i_1 < i_2 < i_3} P(B_{i_1} B_{i_2} B_{i_3}) - \dots + (-1)^9 P(B_1 B_2 \dots B_8),$$

and from Ross [pp. 44–45], we know that this is equal to $1 - \frac{1}{2!} + \frac{1}{3!} - \dots - \frac{1}{8!}$. Therefore,

$P(B) = 1 - P(B^c) = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{1}{8!} \approx 0.36788$. We know, from Ross [pp. 44–45], that as the number of stations gets large, this number converges to $e^{-1} \approx 0.36788$ (it appears that 8 is large enough!). It is further easy to see that the probability that station A gets the wrong packet, $P(A) = 7/8$. Therefore, the desired probability

$$P(B | A) = P(B)/P(A) \approx 0.36788 \times 8/7 \approx 0.4204$$

4. Frantic search

Let D, T, B , and O denote the events that the glasses are in the drawer, on the table, in the briefcase, or in the office, respectively. These four events partition the probability space.

(a) Let E denote the event that the glasses were not found in the first drawer search.

$$P[T|E] = \frac{P[TE]}{P[E]} = \frac{P[E|T]P[T]}{P[E|D]P[D] + P[E|D^c]P[D^c]} = \frac{(1)(0.06)}{(0.1)(0.9) + (1)(0.1)} = \frac{0.06}{0.19} \approx 0.315$$

(b) Let F denote the event that the glasses were not found after the first drawer search and first table search.

$$P[B|F] = \frac{P[BF]}{P[F]} = \frac{P[F|B]P[B]}{P[F|D]P[D] + P[F|T]P[T] + P[F|B]P[B] + P[F|O]P[O]} = \frac{(1)(0.03)}{(0.1)(0.9) + (0.1)(0.06) + (1)(0.03) + (1)(0.01)} \approx 0.22$$

(c) Let G denote the event that the glasses were not found after the two drawer searches, two table searches, and one briefcase search.

$$P[O|G] = \frac{P[OG]}{P[G]} = \frac{P[G|O]P[O]}{P[G|D]P[D] + P[G|T]P[T] + P[G|B]P[B] + P[G|O]P[O]} = \frac{(1)(0.01)}{(0.1)^2(0.9) + (0.1)^2(0.06) + (0.1)(0.03) + (1)(0.01)} \approx 0.4424$$

5. Let's make a deal

(a.i) Assume Dilbert is using the “stick to original guess” strategy. He will win if and only if his original selection of curtain contains the prize. Since the curtains are selected with equal probability, the probability of winning is $\frac{1}{3}$ for this strategy.

(a.ii) Assume Dilbert is using the “switch” strategy. Given his original choice of curtain contains the prize, his conditional probability of winning is zero. Given his original choice of curtain does not contain the prize, his conditional probability of winning is one. So $P[W] = 0 \times \frac{1}{3} + 1 \times \frac{2}{3} = \frac{2}{3}$. That is, Dilbert wins with probability $\frac{2}{3}$ if he uses the “switch” strategy.

(b.i) Assume Dilbert is using the “no switch” strategy. He will win if and only if his original selection of curtain contains the prize. Since the curtains are selected with equal probability, the probability of winning is $\frac{1}{3}$ for the no switch strategy.

(b.ii) Assume Dilbert is using the “switch” strategy. The probability space can be partitioned into three events, each with probability $\frac{1}{3}$: Either: Dilbert's original choice is the curtain with the prize, Wally's original choice is the curtain with the prize, or neither originally choose the curtain with the prize. Given that either of the first two of these events happens, Dilbert's chance of winning is 0. Given the third of these events, Dilbert's probability of winning is 1/2 (this is the probability that Wally, rather than Dilbert, is eliminated, given that both originally select junk curtains). So Dilbert's probability of winning under the “switch” strategy is $\frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$.

6. Circuit relay

Let E denote the event that an A-B path exists and let E_i denote the event that the i^{th} switch is closed.

$$(a) P[E] = P[(E_1 \cup E_2)(E_3 \cup E_4)E_5] = P[E_1 \cup E_2]P[E_3 \cup E_4]P[E_5]$$

$$= (p_1 + p_2 - p_1 p_2)(p_3 + p_4 - p_3 p_4)p_5.$$

$$(b) P[E] = P[E|E_1 E_3]P[E_1 E_3] + P[E|(E_1 E_3)^c]P[(E_1 E_3)^c]$$

$$= 1 \cdot p_1 p_3 + P[E|(E_1 E_3)^c](1 - p_1 p_3)$$

$$= p_1 p_3 + P[E_2(E_4 \cup E_5)](1 - p_1 p_3)$$

$$= p_1 p_3 + p_2(p_4 + p_5 - p_4 p_5)(1 - p_1 p_3)$$

7. Random selection of a card

(a.i) Take S to be the set of all $52!$ permutations of the cards. Let F_i denote the event that the first ace is in the i th position and A denote the event that the 2 of Clubs is in the position following the first ace. We need to calculate $P[A|F_4]$. But $P[A|F_4] = \frac{|AF_4|}{|F_4|}$. Now $|F_4| = 4 \cdot 48 \cdot 47 \cdot 46 \cdot 48!$ because there are 4 choices for which ace to put in position 4, and then 48 choices of position for the next ace, 47 for the next, 46 for the last ace, and $48!$ for the other 48 cards.

Also $|AF_4| = 4 \cdot 47 \cdot 46 \cdot 45 \cdot 47!$ because because the Two of Clubs must go in the 5^{th} position, there are 4 choices for which ace to put in position 4, then 47 choices of position for the next ace, 46 for the next, 45 for the last ace, and $47!$ for the other 47 cards.

Thus $P[A|F_4] = \frac{4 \cdot 47 \cdot 46 \cdot 45 \cdot 47!}{4 \cdot 48 \cdot 47 \cdot 46 \cdot 48!} = \frac{45}{48^2} \approx 0.01953$.

(a.ii) Similarly, $P[A|A_{30}] = \frac{19}{22 \cdot 48} \approx 0.1799$. (Note: One can derive, as in part (a.i), that $P[A|F_i] = \frac{49-i}{(52-i) \cdot 48}$.

A simple explanation for this simple answer is that given F_i is true, the probability the Two of Clubs will be after the first ace is $\frac{49-i}{48}$, and given F_i and given the Two of Clubs is after the first ace, the conditional probability that it is in the first of the $52 - i$ positions after the first ace is $\frac{1}{52-i}$.

(a.iii) [See Ross, Example 5j, Chapter 2] Any permutation in S can be obtained as follows. Take the Two of Clubs out of the deck. There are $51!$ orders for the remaining cards. For each such ordering, there are 52 possible places to insert the Two of Clubs. The event A is true for one out of the 52 choices, namely if the Two of Clubs is inserted into the deck just after the first ace. So $P[A] = \frac{1}{52}$.

(b.i) Let B denote the event that the Ace of Spades follows the first ace, and let the F_i be the events defined above. $|BF_4| = 3 \cdot 47 \cdot 46 \cdot 48!$ since the Ace of Spades must go in the fifth position, there are 3 choices of which ace to put in position 4, 47 choices of where to put the third ace, 46 choices of where to put the fourth ace, and $48!$ choices to put the other 48 cards.

So $P[BF_4] = \frac{|BF_4|}{|F_4|} = \frac{3 \cdot 47 \cdot 46 \cdot 48!}{4 \cdot 48 \cdot 47 \cdot 46 \cdot 48!} = \frac{3}{4 \cdot 48} \approx 0.01562$.

(b.ii) Similarly, $P[B|F_{30}] = \frac{3}{4 \cdot 22} \approx 0.03409$. (Note: One can derive, as in part (b.i), that $P[A|F_i] = \frac{3}{4 \cdot (52-i)}$.

A simple explanation for this simple answer is that given F_i is true, the probability the first ace is not the Ace of Spades is $\frac{3}{4}$, and given F_i and given the first ace is not the Ace of Spades, the conditional probability that the Ace of Spades in the first of the $52 - i$ positions after the first ace is $\frac{1}{52-i}$.

(b.iii) [See Ross, Example 5j, Chapter 2] Any permutation in S can be obtained as follows. Take the Ace of Spades out of the deck. There are $51!$ orders for the remaining cards. For each such ordering, there are 52 possible places to insert the Ace of Spades. The event A is true for one out of the 52 choices, namely if the Ace of Spades is inserted into the deck just after the first of the other three aces. So $P[B] = \frac{1}{52}$.