

1. Express each of the following events in terms of the events A, B , and C , and the operations of complementation, union, and intersection:

- (a) at least one of the events A, B, C occurs; — $A \cup B \cup C$
- (b) at most one of the events A, B, C occurs; — $(AB^cC^c) \cup (A^cBC^c) \cup (A^cB^cC) \cup (A^cB^cC^c)$
- (c) none of the events A, B, C occurs; — $(A \cup B \cup C)^c = A^cB^cC^c$
- (d) all three events A, B, C occur; — ABC
- (e) exactly one of the events A, B, C occurs; — $(AB^cC^c) \cup (A^cBC^c) \cup (A^cB^cC)$
- (f) events A and B occur, but not C ; — $ABC^c = AB - C$
- (g) either event A occurs, or if not then B also does not occur. — $A \cup A^cB^c$

Note — there are many other possible ways to express these answers (think back to ECE 290). For example, for part (b) $(B^cC^c) \cup (A^cC^c) \cup (A^cB^c)$ and for part (g) $A \cup B^c$.

2. Eight islanders are in competition for a large sum of money. Some of the islanders consider the possibility of forming alliances to improve their chances of winning.

- (a) Only one alliance is formed, and it contains at least two people. How many such alliances are there?
The easy way is to compute the number of all subsets of eight islanders (there are 2^8 of these), and to then subtract the number subsets that contain less than two people (eight subsets of size one and one subset of size zero): $2^8 - 8 - 1 = 247$. Another way is to count all subsets of size 2, 3, ... 8

$$\begin{aligned}
 N &= \sum_{i=2}^8 \binom{8}{i} \\
 &= \binom{8}{2} + \binom{8}{3} + \binom{8}{4} + \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8} \\
 &= 28 + 56 + 70 + 56 + 28 + 8 + 1 = 247
 \end{aligned}$$

- (b) How many of these possible alliances contain Rich?

In order to form a valid alliance (i.e., at least two people), Rich needs to join with at least one other person from the remaining seven. Thus, we can simply compute the number of subsets of the seven remaining islanders that contain at least one person. Each such subset, when joined with Rich, forms a valid alliance. Thus, there are $2^7 - 1 = 127$ alliances that contain Rich.

- (c) Two of the islanders, Kelly and Sue, don't really like one another, and refuse to join the same alliance. Once again, one alliance is formed, and it contains at least two people (but not both of Kelly and Sue). How many such alliances can be formed?

There are three ways such an alliance can be formed: neither Kelly or Sue are in the alliance (the alliance must contain at least two people from the other six islanders); Kelly and at least one member of the other six islanders; Sue and at least one member of the other six islanders. Following reasoning analogous to that above, the number of alliances in these three categories is: $2^6 - 6 - 1 = 57$, $2^6 - 1 = 63$, and $2^6 - 1 = 63$, for a total of 183 possible alliances that do not contain both Kelly and Sue.

A possibly easier way to compute the answer is to note that of the possible alliances, 2^6 contain both Sue and Kelly (i.e., Sue, Kelly and *any* subset of the remaining six islanders). Using the results from part (a) $247 - 64 = 183$.

- (d) How many of these alliances that do not contain both Sue and Kelly do contain Rich?

We can proceed as above by enumerating the three cases: Rich and at least one person, not including either Kelly or Sue; Rich and at least one person, one of which is Kelly; Rich and at least one person, one of which is Sue. Thus we have $2^5 - 1 = 31$, $2^5 = 32$, and $2^5 = 32$, for a total of 95 such alliances. Again, we could also solve this problem by noting that there are 127 alliances that contain Rich, and that of these, 2^5 contain both Sue and Kelly (i.e., Rich, Kelly, Sue and *any* subset of the remaining five islanders). Thus, as in part (c), we obtain $127 - 32 = 95$.

- (e) If the islanders form two disjoint alliances of four people each, a deadlock will occur. How many ways can this occur?

Call the alliances Alliance A and Alliance B. There are 8-choose-4 possible ways to choose Alliance A, and each of these defines a corresponding Alliance B. However, half of these are redundant. For

example, $A = \{\text{Sue, Rich, Colleen, Gervase}\}$ and $B = \{\text{Kelly, Rudy, Greg, Sean}\}$ is equivalent to $A = \{\text{Kelly, Rudy, Greg, Sean}\}$ and $B = \{\text{Sue, Rich, Colleen, Gervase}\}$ in terms of creating a deadlock. Therefore, the number of deadlock situations is

$$\frac{1}{2} \binom{8}{4} = 35.$$

3. Continuing with the above scenario, each of the eight islanders will cast one vote for either Kelly or Rich.

- (a) How many possible results are there for the voting? (Note: a result is defined as the pair (n_r, n_k) , where n_r is the number of votes for Rich and n_k is the number of votes for Kelly.)

The possible results are $(0, 8), (1, 7), \dots, (8, 0)$. Thus there are 9 possible outcomes.

- (b) Sue states that she will not vote for Kelly. Now how many possible results are there? The possible results are now $(1, 7), (2, 6), \dots, (8, 0)$. Thus there are 9 possible outcomes.

- (c) Rich wins if he receives at least five votes. How many assignments of votes to islanders produce a win for Rich?

Rich wins is a subset of at least five islanders vote for him. Thus we can simply sum the number of subsets of size 5, 6, 7, and 8:

$$N = \sum_{i=5}^8 \binom{8}{i} = \binom{8}{5} + \binom{8}{6} + \binom{8}{7} + \binom{8}{8} = 56 + 28 + 8 + 1 = 93$$

- (d) How many assignments of votes to islanders produce a tie (i.e., four votes for Kelly and four votes for Rich)?

This question is equivalent to the question of how many subsets of size four exist, yielding 8-choose-4, or 70 possible ways to produce the tie.

4. A large urn contains seven slips of paper. The name *Rich* is written on four of the slips, and the name *Kelly* is written on remaining three. The slips are drawn at random, one by one until the urn is empty.

- (a) In how many possible orders can the two names be drawn?

If the slips were each uniquely identifiable, there would be $7!$ orderings. Here, four slips are lumped into an equivalence class (Rich), with $4!$ possible orderings, and three others are lumped into another equivalence class (Kelly), with $3!$ possible orderings. Thus, the total is

$$\frac{7!}{4!3!} = 35.$$

Another way to solve this problem is to note that the four *Rich* slips will appear in four positions ($7\text{-choose-}4 = 35$), and once these are determined, the positions of the *Kelly* slips are fixed.

- (b) A dramatic finish occurs when *Rich* is the last name drawn. What is the probability of a dramatic finish?

For a dramatic finish, the last item must be Rich. Thus the question can be formulated as: in how many possible orders can the six slips, three with *Kelly* and three with *Rich*, be drawn. The reasoning is analogous to that above,

$$\frac{6!}{3!3!} = 20.$$

And therefore the probability is $20/35 = 4/7$. Note that this is consistent with the result obtained using *sampling without replacement* (see the Powerpoint slides on the web), which states that the probability that *Rich* will be drawn on any of the draws is equal to $4/7$.

- (c) What is the probability that *Rich* will be drawn on each of the first four draws?

Consider all possible (i.e., $7!$) orderings of the slips. Of these, there are $4!$ possible ways to draw the first four slips with *Rich*, and for each such ordering of the first four slips there are $3!$ orderings for the last three slips. Therefore the probability is $4!3!/7! = 1/35$. Alternatively, one could reason that there are $7\text{-choose-}4$ (i.e., 35) ways to draw the first four names. Of these, one gives *Rich* for the first four - thus, $1/35$ is the probability.

5. At the tribal council, seven islanders sit in a row. How many ways can the islanders be seated for each of the following conditions?

- (a) There are no constraints on seating order.

There are $7! = 5040$ possible orderings.

- (b) Rudy and Rich must be seated next to one another.

Suppose that Rudy sits immediately to Rich's left. Then there are six possible seats for Rudy. If Rudy sits immediately to Rich's right, there are six possible seats for Rudy. Thus, there are twelve possible choices for seating Rudy and Rich. For each of these twelve choices, the remaining five islanders can be seated in any order in the remaining empty seats. Thus, the number of orderings is $2 \times 6 \times 5! = 1440$.

- (c) No two of Rudy, Sue, Kelly or Gervase may be seated next to one another.

Since there are only seven seats, this constraint implies that these four must be seated in the odd numbered seats. There are $4!$ orderings for these four once they are confined to the four odd numbered seats. For each such ordering, there are $3!$ orderings for the remaining three. Thus, the number of orderings is $4!3! = 144$.

- (d) Colleen and Greg must not be seated next to each other.

There are two cases to consider: Colleen sits in one of the two end seats (in which case there are five choices for Greg's seat), or Colleen sits in one of the five non-end seats (in which case Greg has only four choices). Once Greg and Colleen have chosen their seats, the remaining islanders can take seats in any order in the remaining five seats. Thus, the number of orders is $(2 \times 5 \times 5!) + (5 \times 4 \times 5!) = 1200 + 2400 = 3600$.

- (e) All of constraints b,c and d must be satisfied.

As above, constraint (c) implies that Rudy, Sue, Kelly and Gervase must sit in the odd numbered seats. With respect to constraint (b), there are *two* interesting cases: Rudy sits in one of the two end seats, or Rudy sits in one of the two non-end seats. If Rudy sits in one of the *two* end seats, then Rich has only *one* choice for his seat, while if Rudy sits in one of the *two* odd non-end seats, then Rich has *two* choices for his seat. Once Rudy has chosen his seat, Sue, Kelly and Gervase are free to choose arbitrarily among the remaining odd numbered seats, giving $3!$ possible orderings for these three. Once Rudy, Rich, Sue, Kelly, and Gervase have been seated, any choice by Colleen and Greg for the remaining *two* seats will satisfy constraint (d), giving $2!$ choices for these two. Therefore, we have $(2 \times 1 \times 3! \times 2!) + (2 \times 2 \times 3! \times 2!) = 24 + 48 = 72$ (note: the first term is for Rudy in an end seat, and the second term is for Rudy in a non-end seat).

6. There are seven supermarkets in a town. Three visitors pick supermarkets at random to shop. Find the probability for each of the following events.

- (a) They all choose the same supermarket.

There are 7^3 possible outcomes in the sample space. There are seven possible ways that they can all choose the same supermarket (they all chose the first, they all choose the second...). Thus,

$$P(\text{all chose the same}) = 7/7^3 = 1/49.$$

- (b) They choose three different supermarkets.

There are $7 \times 6 \times 5$ ways that they can choose three different supermarkets. Thus,

$$P(\text{three different supermarkets}) = \frac{7 \times 6 \times 5}{7^3} = 30/49$$

- (c) They do not all choose the same supermarket.

This one is easy given the answer to part (a):

$$P(\text{not all choose the same}) = 1 - P(\text{all chose the same}) = 1 - 1/49 = 48/49.$$

- (d) At least two of them choose the same supermarket.

Again, this one is easy using earlier results.

$$P(\text{at least two choose the same}) = 1 - P(\text{three different supermarkets}) = 1 - 30/49 = 19/49.$$

7. An experiment consists of picking a student from the set of all students registered on the UIUC campus this semester. It is *not* necessary to assume that all students are equally likely to be picked, but you may make this assumption if you like.

- (a) Consider the two events:

- A — the student has had four years of high school science (FYS),
- B — the student has had calculus in high school.

If the probability that the student has had at least one of FYS and calculus is 0.6, and the probability that the student has missed at least one of the two is 0.8, what is the probability that the student has had *exactly* one of the two?

Let $E = A \cup B$ and $F = A^c \cup B^c$. We are given $P(E) = 0.6$ and $P(F) = 0.8$. Furthermore, note that $E \cup F = \Omega$. The event that corresponds to exactly one of FYS or calculus is the exclusive or $A \oplus B = AB^c \cup A^cB = E \cap F$. Given these relationships, we have

$$P(E \cup F) = P(E) + P(F) - P(EF) \rightarrow P(EF) = 0.4$$

- (b) Let C denote the event that the student is registered in ECE313 this semester, and let events A and B and their probabilities be as in part (a). If students who have had at most one of FYS and calculus did not register in ECE313 this semester,

- i. What is $P(A^c \cap B^c \cap C^c)$?

Using DeMorgan's laws, we obtain $P(A^cB^c) = 1 - P(A \cup B) = 1 - P(E) = 0.4$. We are given that $C \cap A^c = C \cap B^c = \emptyset \rightarrow A^cB^cC = \emptyset \rightarrow P(A^c \cap B^c \cap C) = 0$. We can partition the event $A^c \cap B^c$ into the disjoint events $A^c \cap B^c \cap C^c$ and $A^c \cap B^c \cap C$. Thus $0.4 = P(A^c \cap B^c) = P(A^c \cap B^c \cap C^c) + P(A^c \cap B^c \cap C) = P(A^c \cap B^c \cap C^c)$.

- ii. What is the probability that the student picked is not registered in ECE313 *and* has had exactly one of FYS or calculus in high school?

The event of interest is $(A \oplus B) \cap C^c$. We can partition the event $(A \oplus B)$ into the disjoint events $(A \oplus B) \cap C^c$ and $(A \oplus B) \cap C$. But, we know that $P((A \oplus B) \cap C) = 0$, since if a student has had exactly one of FYS or calculus the student cannot be in ECE 313. Thus, using the results from part (a), $0.4 = P(A \oplus B) = P((A \oplus B) \cap C^c) + P((A \oplus B) \cap C) = P((A \oplus B) \cap C^c)$.

- (c) Using the data given in parts (a) and (b), which of the following probabilities:

$$P(ABC), P(C), P(A^cBC^c), P(A), P(A^cB^cC), P(A \cup B \cup C^c)$$

can you compute? It is not necessary to actually compute each probability.

From part (b) we have $P(A^cB^cC) = 0$. Using DeMorgan's laws we have $P(A \cup B \cup C^c) = 1 - P(A^cB^cC) = 1$. The probabilities $P(ABC)$, $P(A)$, $P(C)$, $P(A^cBC^c)$ cannot be computed.