

1. Geometric series

(a) Since $x \neq 1$, the equation to be proved is equivalent to the equation we get by multiplying both sides by $(1 - x)$. So it is sufficient to prove that $(1 - x)(1 + x + x^2 + \dots + x^{n-1}) = 1 - x^n$. But

$$\begin{aligned}(1 - x)(1 + x + x^2 + \dots + x^{n-1}) &= (1 + x + x^2 + \dots + x^{n-1}) - x(1 + x + x^2 + \dots + x^{n-1}) \\ &= (1 + x + x^2 + \dots + x^{n-1}) - (x + x^2 + x^3 + \dots + x^n) \\ &= 1 - x^n,\end{aligned}$$

which establishes the desired equation. (Another way to prove the equation is by induction on n .)

(b) Since $\lim_{n \rightarrow \infty} x^n = 0$ under the assumption $|x| < 1$, taking the limit as $n \rightarrow \infty$ in both sides of the equation in part (a) yields that $1 + x + x^2 + \dots = \frac{1}{1-x}$.

2. Inverse function

Sketching f , we see that f is monotonically decreasing, with limit 2 at $-\infty$ and limit 0 at ∞ . In particular, an inverse function exists since f is one-to-one. The range of f is $(0, 2)$, so the inverse g needs only to be defined on that interval. We need $f(g(u)) = u$ for $0 < u < 2$. Solving this equation for $g(u)$ yields $\frac{2}{e^{g(u)} + 1} = u$, or $\frac{2}{u} - 1 = e^{g(u)}$, or $g(u) = \ln(\frac{2}{u} - 1)$ for $0 < u < 2$.

3. Extrema of functions

(a) The function $f(x) = x^4(1.001)^{-x^2}$ can be written in the form $x^n \exp(-ax^2)$, where $a = \ln(1.001)$, $n = 4$. The function $x^n \exp(-ax^2)$ has local extrema at the points $x = \pm\infty, 0, \pm\sqrt{n/2a}$. In this case $n = 4$ and therefore the maxima occur at $x = \pm\sqrt{n/2a} = \pm 44.7325$ (as can be verified by differentiating once again and checking the sign of the result at the two points); the points $x = 0, \pm\infty$ represent minima. The maximum function value (at both maxima) is 5.4188×10^5 .

(b) If $f(x) = x^4(1.001)^{x^2}$, the function converges to infinity as $|x| \rightarrow \infty$. Thus, $f(x)$ is not a maximum value of f for any finite value of x . Incidentally, carrying out the differentiation gives us $x = \pm\sqrt{-2/\ln(1.001)}$ as the other solutions, which is not possible since the quantity within the “ $\sqrt{}$ ” sign is negative.

4. Some definite integrals

(a) Drawing a picture, you see that the integral from -1 to 1 is just the sum of the areas of two triangles, each having height 1 and base length 1. Therefore, it is easy to see that the total area $= 2 \times 1/2 = 1$. Try and draw a little sketch before you tackle most problems. Those who used formal integration principles should note that $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. Therefore

$$\int_{-1}^1 |x| dx = \int_{-1}^0 -x dx + \int_0^1 x dx = \left. \frac{-x^2}{2} \right|_{-1}^0 + \left. \frac{x^2}{2} \right|_0^1 = 1.$$

(b) Substitute $y = (1 - x^2)$. Then

$$\int_0^1 x(1 - x^2)^{11} dx = -\frac{1}{2} \int_1^0 y^{11} dy = \frac{1}{2} \int_0^1 y^{11} dy = \frac{1}{24}.$$

(c) Integrate by parts twice:

$$\begin{aligned}\int_0^1 x^2 e^{-x} dx &= -x^2 e^{-x} \Big|_0^1 + \int_0^1 2x e^{-x} dx \\ &= -e^{-1} + \int_0^1 2x e^{-x} dx \\ &= -e^{-1} - 2x e^{-x} \Big|_0^1 + \int_0^1 2e^{-x} dx \\ &= -3e^{-1} + \int_0^1 2e^{-x} dx \\ &= -3e^{-1} + 2(1 - e^{-1}) = 2 - 5e^{-1} \approx 0.1602\end{aligned}$$

(d) Notice that the integrand is an odd function and therefore it has to integrate to 0 over any interval that is symmetric about the origin. In particular, the integral over the whole real line is zero.

5. Derivatives and integrals

- (a) True as per chain rule.
- (b) Also true as per chain rule.
- (c) False. Chain rule gives $\frac{d}{dx} \exp(f(x^2))$ as $2x \cdot g(x^2) \exp(f(x^2))$.
- (d) False. $\int g(-x) dx = -f(-x) + C$ (minus sign missing.)
- (e) False. The antiderivative of $g(x^2/2)$ need not be related to $f(x^2/2)$ at all, as can be seen by differentiating both sides of (v). For example, take $f(x) = \ln(1+x)$ to see this.
- (f) True only for positive functions $f(x)$.

6. Double integrals

- (a) $f(x, y) = \max(x, y)$, over the region $D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$.

Drawing a three-dimensional picture is not the easiest thing, but it is worth the effort to get an idea of what the function looks like. Applying basic integration techniques and using the fact that $\max(x, y) = y$ if $x \leq y$ and $\max(x, y) = x$ if $x > y$, we get

$$\begin{aligned} \int_0^1 \int_0^1 \max(x, y) dx dy &= \int_0^1 \left(\int_0^y y dx \right) dy + \int_0^1 \left(\int_y^1 x dx \right) dy \\ &= \int_0^1 y^2 dy + \int_0^1 \frac{1-y^2}{2} dy = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}. \end{aligned}$$

- (b) $f(x, y) = (x^2 + y^2)^{-4}$, over the region $D = \{(x, y) : x^2 + y^2 > 1\}$.

Changing to polar coordinates ($x = r \cos \theta, y = r \sin \theta$), the above integral can be reduced to

$$\iint_D (x^2 + y^2)^{-4} = \int_1^\infty \int_0^{2\pi} r^{-8} \cdot r dr d\theta = 2\pi \int_1^\infty r^{-7} dr = \pi/3,$$

where the two-dimensional differential $dx dy$ transforms to $r dr d\theta$.

7. An application of Taylor's approximation

- (a) $f(0) = 0$ and $f'(0) = 1$, so $P(x) = x$.
- (b) $|f''(x)| = |\frac{1}{(1+x)^2}|$ which is monotonically decreasing on the interval $[-0.5, 0.5]$. Thus, the maximum of $|f''|$ over the interval is $|f''(-0.5)| = 4$. Therefore, we can take $M = 4$. (Any larger value of M would also work.)
- (c) Note that $n \ln(1 + \frac{a}{n}) = n f(\frac{a}{n}) \approx n P(\frac{a}{n}) = a$. That is, the limit is a . To do a more careful job, we can bound the rate of convergence as follows. If $n \geq 2a$ then $|\frac{a}{n}| \leq 0.5$ so Taylor's Inequality with $M = 4$ yields

$$\begin{aligned} |n \ln(1 + \frac{a}{n}) - a| &= n |f(\frac{a}{n}) - P(\frac{a}{n})| \\ &\leq n \frac{M}{2!} (\frac{a}{n})^2 \quad \text{if } n \geq 2a \\ &= \frac{2a^2}{n} \end{aligned}$$

- (d) Using the hint,

$$\begin{aligned} (1 + \frac{a}{n})^n &= \exp(\ln((1 + \frac{a}{n})^n)) \\ &= \exp(n \ln(1 + \frac{a}{n})) \xrightarrow{(1)} \exp(a) \end{aligned}$$

The limit (1) is justified because the argument of the exp function converges as shown in part (c), and the exp function is continuous.