

More continuous random variables

Assigned reading: Review *Ross*, Chapter 5 and Section 9.1

Noncredit exercises: Chapter 5, Problems 15-38 and Theoretical Exercises 12-16,21,26,28-30

Reminder: Hour Exam II: Monday, November 11, 7:00-8:00 p.m., Room 269 Everitt Lab. This exam covers the material of problem sets 1-10 (with emphasis on problem sets 6-10) and the associated reading (*Ross* through Chapter 5 and the *Notes on Decision Making* distributed with problem set 5, and lectures through November 4. You may bring one 8.5" by 11" sheet of notes to the hour exam. You may use both sides of the sheet, using font size 10 or larger (or similar handwriting size). The exam is closed book otherwise. Calculators, laptop computers, tables of integrals, etc. are not permitted.

1. Generation of random variables with specified probability density function

Suppose you are writing a simulation program, and you'd like to generate a random variable X with the following probability density function:

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{else.} \end{cases}$$

A call to the random number generator produces a variable U that is uniformly distributed on the interval $[0, 1]$. (At least this is true in theory, in reality computers use a "pseudo-random" number generator.) By applying a function g to U , a new random variable $X = g(U)$ can be produced. Find the increasing function g on the interval $[0, 1]$ such that $X = g(U)$ has the specified probability density. (Hint: Begin by finding the cumulative distribution function F_X .)

2. Normal approximation to the binomial distribution

Do problem 20, p. 230 of *Ross* using the binomial approximation to the normal distribution.

3. The Poisson process

Consider a Poisson process on the interval $[0, T]$ with arrival rate $\mu > 0$, and let $0 < \tau < T$. Define N_1 to be the number of arrivals during $[0, \tau]$, N_2 to be the number of arrivals during $[\tau, T]$, and N to be the total number of arrivals during $[0, T]$. Let i, j, n be nonnegative integers such that $n = i + j$. Express the following probabilities in terms of n, i, j, τ, T , and μ , simplifying your answers as possible:

- (a) $P[N = n]$, (b) $P[N_1 = i]$, (c) $P[N_2 = j]$, (d) $P[N_1 = i | N = n]$, (e) $P[N = n | N_1 = i]$.

4. Waiting times for a Poisson process

Calls arrive to a cell in a certain wireless communication system according to a Poisson process with arrival rate $\mu = 2$ calls per minute. Measure time in minutes and consider an interval of time beginning at time $t = 0$. Let $N(t)$ denote the number of calls that arrive up until time t .

- (a) What is the discrete probability mass function of $N(t)$ for a fixed $t > 0$?

- (b) Find the probabilities of each of the events:

E_1 = "No calls arrive in the first 3.5 minutes"

E_2 = "The first call arrives after time $t = 3.5$ "

E_3 = "2 or fewer calls arrive in the first 3.5 minutes"

E_4 = "the 3rd call arrives after time $t = 3.5$ "

E_5 = "the 3rd call arrives after time t " (For general $t > 0$) (use only finitely many terms)

E_6 = "the 3rd call arrives before time t " (for general $t > 0$) (Hint: $P[E_6] = 1 - P[E_5]$)

- (c) Note that, as a function of t , $P[E_6]$ is the cumulative probability distribution function for the time of the third call arrival. Differentiate this function with respect to t to find the probability density function for the time of the third arrival. Terms cancel, so simplify your answer. You should get the gamma density with parameters $(3, \mu)$.

- (d) What is the expected arrival time of the 10th call?

5. Hypothesis testing for the variance of a Gaussian variate

Based on a sensor measurement X , it has to be decided which hypothesis about a remotely monitored machine is true: H_0 : the machine is working vs. H_1 : the machine is broken. Suppose that if hypothesis H_0 is true then X is normal with mean 0 and variance a^2 , and if H_1 is true then X is normal with mean 0 and variance b^2 . Suppose that $b > a > 0$. Write $f_i(x)$ as the probability density function of X in case H_i is the true hypothesis.

- (a) Compute the likelihood ratio $\Lambda(x) = \frac{f_1(x)}{f_0(x)}$ for the observation x , $-\infty < x < \infty$. Simplify your answer.
- (b) The maximum likelihood decision rule is a likelihood ratio test with threshold $\tau = 1$: it decides that H_1 is true if $\Lambda(X) \geq 1$, and decides H_0 is true otherwise. Show that this rule is equivalent to deciding H_1 is true if $|X| \geq K$ for some threshold K , and find K in terms of a^2 and b^2 .
- (c) If prior probabilities π_0 and π_1 are given for the hypotheses, the MAP decision rule is a likelihood ratio test with threshold $\tau = \frac{\pi_0}{\pi_1}$. Show that the MAP rule is equivalent to deciding H_1 is true if $|X| \geq K$ for some threshold K , and find K in terms of a^2 , b^2 , and the prior probabilities.
- (d) Find the conditional error probabilities, $p_{\text{false_alarm}}$ and p_{miss} for either of the rules above. Express your answers in terms of a , b , K , and either the cumulative distribution function for a standard normal variate Φ or the Q function defined by $Q(u) = 1 - \Phi(u)$. (Here K depends on which rule is used, but the value of K need not concern you in part (d).)

6. Hypothesis testing for the mean of a Laplacian variate.

Based on a sensor measurement X , it has to be decided which hypothesis about a remotely monitored room is true: H_0 : the room is empty vs. H_1 : a person is present in the room. Suppose that if hypothesis H_0 is true then X has pdf $f_0(x) = \frac{1}{2}e^{-|x+1|}$ and if H_1 is true then X has pdf $f_1(x) = \frac{1}{2}e^{-|x-1|}$. Both densities are defined on the whole real line.

- (a) Carefully sketch the two pdfs. (Hint: consider the regions $x < -1$, $-1 < x < 1$ and $x > 1$ separately.)
- (b) Compute the likelihood ratio $\Lambda(x) = \frac{f_1(x)}{f_0(x)}$ for the observation x , $-\infty < x < \infty$. Simplify your answer.
- (c) The maximum likelihood decision rule is a likelihood ratio test with threshold $\tau = 1$: it decides that H_1 is true if $\Lambda(X) \geq 1$, and decides H_0 is true otherwise. Show that this rule is equivalent to deciding H_1 is true if $X \geq K$ for some threshold K , and find K .
- (d) Find the conditional error probabilities, $p_{\text{false_alarm}}$ and p_{miss} for the maximum likelihood rule.
- (e) Suppose prior probabilities $\pi_0 = 2/3$ and $\pi_1 = 1/3$ are assumed for the hypotheses. The MAP decision rule is a likelihood ratio test with threshold $\tau = \frac{\pi_0}{\pi_1} = 2$. Show that the MAP rule is equivalent to deciding H_1 is true if $X \geq K$ for some threshold K , and find K in terms of the prior probabilities.
- (f) Find the conditional error probabilities, $p_{\text{false_alarm}}$ and p_{miss} for the MAP decision rule of part (e).