

**Continuous Random Variables**  
Expectation, Variance, and Uniform Random Variables

**Assigned reading:** *Ross* Chapter 5, Sections 1-3.  
**Noncredit Exercises:** pp. 228-231: 1-4, 6-8, 10-14.

**1. Properties of probability density functions.**

For each of the following, indicate whether the specified function is a valid probability density function. Assume that the functions are zero outside of the specified ranges. For those functions that are not valid pdfs, state at least one property of pdfs that is not satisfied. Also, state whether there exists a constant  $C$  such that  $Cf(x)$  is a valid pdf even though  $f(x)$  is not.

- (a)  $f(u) = |u|$  for  $|u| < 1$ .
- (b)  $f(u) = 1 - |u|$  for  $|u| < 1$ .
- (c)  $f(u) = \ln u$  for  $0 < u < 1$ .
- (d)  $f(u) = \ln u$  for  $0 < u < 2$ .
- (e)  $f(u) = 2u$  for  $0 < u < 1$ .
- (f)  $f(u) = \frac{2}{3}(u - 1)$  for  $0 < u < 3$ .
- (g)  $f(u) = \exp(-2u)$  for  $0 < u < \infty$ .
- (h)  $f(u) = 4\exp(-2u) - \exp(-u)$  for  $0 < u < \infty$ .

**2. Computing probabilities using density functions.**

Let  $\mathbf{X}$  denote a continuous random variable with pdf  $f_{\mathbf{X}}(x)$  given by

$$f_{\mathbf{X}}(x) = \begin{cases} 1+x & : -1 < x \leq 0 \\ x & : 0 < x < 1 \\ 0 & : \text{else} \end{cases}$$

- (a) Find  $P\{|\mathbf{X}| < \frac{1}{2}\}$
- (b) Find  $P\{\mathbf{X} > 0 \mid |\mathbf{X}| < \frac{1}{2}\}$
- (c) Find  $E[\mathbf{X}]$ .
- (d) Find  $E[|\mathbf{X}|]$ .

**3. Expectation of functions of random variables I.**

Let  $\mathbf{X}$  be a uniform random variable on the interval  $[-1, 1]$ .

- (a) For  $\mathbf{Y} = \mathbf{X}^2$ , compute the mean and variance of  $\mathbf{Y}$ .
- (b) Compute  $E[\mathbf{Z}]$  for  $\mathbf{Z} = g(\mathbf{X})$  with

$$g(x) = \begin{cases} x^2 & : x \geq 0 \\ -x^2 & : x < 0 \end{cases}$$

#### 4. Expectation of functions of random variables II.

Consider a sphere whose radius is a random variable,  $R$ , with pdf  $f_R(u) = 2u$  for  $0 < u < 1$ .

- (a) What is the average radius of the sphere?
- (b) What is the average volume of the sphere? If we call a sphere with average radius an average sphere, then does the average sphere have the average volume?
- (c) What is the average surface area of the sphere? Does the average sphere have the average surface area?
- (d) Show that  $E[R] > E[R^2] > E[R^3]$  if the pdf for  $R$  is any valid pdf that is nonzero only on the unit interval  $(0, 1)$ .

#### 5. The economics of supply and demand.

Let  $\mathbf{X}$  be a random variable that denotes the weekly demand (measured in thousands of gallons) for gasoline at a particular gas station. The pdf for  $\mathbf{X}$  is given by

$$f_{\mathbf{X}}(x) = \begin{cases} 5(1-x)^4 & : 0 < x < 1 \\ 0 & : \text{else} \end{cases}.$$

Let  $C$  denote the capacity (in thousands of gallons) of the station's storage tank, which is refilled weekly. Let  $\mathbf{Y}$  be a random variable that denotes the amount (measured in thousands of gallons) of gasoline sold in a week. Note that the amount of gasoline sold cannot exceed the tank's capacity, regardless of the demand. Finally, assume that the gross profit for each gallon sold is \$0.64.

- (a) Suppose that  $C = 0.5$  and for a particular week the random variable  $\mathbf{X}$  takes the value 0.68. Can the gas station satisfy the demand?
- (b) Suppose that  $C = 0.5$  and for a particular week the random variable  $\mathbf{X}$  takes the value 0.43. Can the gas station satisfy the demand?
- (c) Suppose that  $C = 0.5$ . What is the probability that the weekly demand for gasoline can be satisfied? Note: if your answer is, e.g., 0.25, then, in the long run, the gas station can supply demand only one out of four weeks.
- (d) What is the minimum value for  $C$  that will ensure the probability that the demand exceeds the supply is no larger than  $10^{-5}$ ?
- (e) How is  $\mathbf{Y}$  related to  $\mathbf{X}$ ?
- (f) What is the expected value of weekly gross profit?
- (g) Suppose that the owner pays \$  $20C$  as a weekly rent on a tank of capacity  $1000C$  gallons (with  $0 \leq C \leq 1$ ). What is the expected value of the weekly net profit? What value of  $C$  will maximize this?