University of Illinois at Urbana-Champaign

# ECE 313: Probability with Engineering Applications

**Fall 2002**
**Final Exam**

**Wednesday, December18, 2002**

Name: ______________________________________________

- You have three hours for this exam. The exam is closed book; however, you may consult both sides of two $8.5'' \times 11''$ sheets of notes.
- Calculators, laptop computers, Palm Pilots, two-way e-mail pagers, etc. may not be used.
- Write your answers in the spaces provided.
- It is *not necessary* to convert answers that are simple common fractions into decimal fractions, but you should simplify the fractions by canceling common factors from the numerator and denominator (e.g., write 5/16 instead of 10/32).
- **Please show all of your work. Answers without appropriate justification will receive very little credit.** If you need extra space, use the back of the previous page.

Score:

1. __________ (12 pts.)
2. __________ (10 pts.)
3. __________ (8 pts.)
4. __________ (16 pts.)
5. __________ (12 pts.)
6. __________ (20 pts.)
7. __________ (16 pts.)
8. __________ (18 pts.)

Total: __________(112 pts.)

**Problem 1** *(12 points)*

Let $A$ and $B$ be arbitrary events with $0 < P(A), P(B) < 1$. Indicate whether each of the following statements is true or false by clearly writing "true" or "false" on the line provided after each letter.

**(a)** ______: $P(A \cup B) \leq P(A) + P(B)$.

**(b)** ______: $P(A|B) \leq 1$.

**(c)** ______: $P(A|B) + P(A|B^c) = 1$.

**(d)** ______: $P(A|B) + P(A^c|B) = 1$.

**(e)** ______: $\frac{P(A|B)}{P(B)} \leq 1$.

**(f)** ______: $P(AB^c) = P(A) - P(AB)$.

**Problem 2** *(10 points)*

Let $X$ and $Y$ be uncorrelated, jointly Gaussian random variables, with parameters $\mu_X$, $\mu_Y$, $\sigma_X^2$ and $\sigma_Y^2$. Indicate whether each of the following statements is true or false by clearly writing "true" or "false" on the line provided after each letter.

**(a)** ______: $f_{XY}(x, y) = f_X(x) f_Y(y)$.

**(b)** ______: $E[XY] = 0$.

**(c)** ______: $Var(X + Y) = \sigma_X^2 + \sigma_Y^2$.

**(d)** ______: $P\{X + Y \leq z\} = \Phi\left(\frac{z - \mu_X - \mu_Y}{\sqrt{\sigma_X^2 + \sigma_Y^2}}\right)$.

**(e)** ______: $P\{X \leq x | Y \leq y\} = \Phi(x)$.

**Problem3** *(8 points)*

Let $X$ and $Y$ be arbitrary random variables, with means $\mu_X$ and $\mu_Y$ and variances $\sigma_X^2$ and $\sigma_Y^2$. Indicate whether each of the following statements is true or false by clearly writing "true" or "false" on the line provided after each letter.

**(a)** ______: If $E[X^2] = E[Y^2]$, then $\sigma_X^2 = \sigma_Y^2$.

**(b)** ______: $Cov(X + Y, X - Y) = \sigma_X^2 - \sigma_Y^2$.

**(c)** ______: If $Var(X + Y) = \sigma_X^2 + \sigma_Y^2$ then $X$ and $Y$ are independent.

**(d)** ______: If $\sigma_X^2 = \sigma_Y^2$ and $\mu_X = \mu_Y = 0$, then $\rho = 1$ (i.e., $X$ and $Y$ are perfectly correlated).

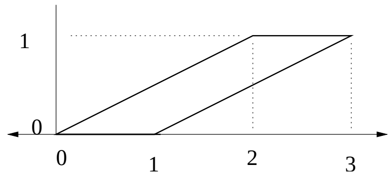


Figure 1:

**Problem 4** *(16 points)* Let the random variables $X$ and $Y$ be jointly uniformly distributed over the region shown.

(a) *(2 points)* Determine the value of $f_{XY}$ on the region shown.

(b) *(2 points)* Find $f_X$, the marginal pdf of $X$.

(c) *(4 points)* Find the mean and variance of $X$.

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(d) *(2 points)* Find the conditional pdf of Y given that $X = a$, for $0 \leq a \leq 1$

(e) *(2 points)* Find the conditional pdf of Y given that $X = a$, for $1 \leq a \leq 2$

(f) *(4 points)* Find and sketch $E[Y|X = x]$ as a function of $x$. Be sure to specify which range of $x$ this conditional expectation is well defined for.

**Problem 5** *(12 points)* A bag contains four oranges and three apples. A child reaches into the bag with both hands and removes two pieces of fruit. Let the random variable $L = 1$ when the fruit in the left hand is an orange and $L = 0$ when the fruit in the left hand is an apple. Similarly, let the random variable $R = 1$ when the fruit in the right hand is an orange and $R = 0$ when the fruit in the right hand is an apple.

(a) *(2 points)* Compute $p_L$, the probability mass function for $L$.

(b) *(3 points)* Compute $p_{LR}$, the joint probability mass function for $L$ and $R$.

(c) *(2 points)* Compute the covariance $Cov(L, R)$.

(d) *(2 points)* Are $L$ and $R$ independent? Explain.

(e) *(3 points)* Compute $p_Z$, the probability mass function for $Z = L + R$.

**Problem 6** *(20 points, 4 points for each part)* The length of time a professor can write on the blackboard without breaking the chalk is a random variable, $T$ (measured in minutes) with pdf given by

$$f_T(t) = \begin{cases} 0.1e^{-0.1t} & : \ t \geq 0 \\ 0 & : \ t < 0 \end{cases}$$

(a) Determine the CDF $F_T$ for $T$.

(b) Determine $E[T]$.

(c) If the professor has not broken the chalk after five minutes, what is the probablility that he will break the chalk in the next ten minutes?

Now, suppose that the professor's chalk breaking can be modeled as a Poisson process with arrival rate 0.1.
(d) What is the probability that the professor will break three pieces of chalk in the first ten minutes of class?

(e) Let $N$ be a random variable that denotes the number of times the professor breaks a piece of chalk. What is the expected value of $N$ if the class is 100 minutes in length?

**Problem 7** *(16 points, 4 points for each part)* On the basis of a sensor output $X$, it is to be decided which hypothesis is true: $H_0$ or $H_1$. Suppose that if $H_0$ is true then $X$ is exponentially distributed with parameter $\mu_0$, and if $H_1$ is true then $X$ is exponentially distributed with parameter $\mu_1$. Assume that $\mu_0 > \mu_1$.

(a) Describe the Maximum Likelihood (ML) decision rule for deciding which hypothesis is true for observation $X$.

(b) Find the probabilty of false alarm, $p_{false\_alarm}$ and the probability of a miss, $p_{miss}$ in terms of $\mu_0$ and $\mu_1$.

(c) Suppose it is known *a priori* that $H_0$ is true with probability $\pi_0$ and $H_1$ is true with probability $\pi_1 = 1 - \pi_0$. For what values of $\pi_0$ does the MAP decision rule always declare that $H_1$ is true, no matter what the value of the observation $X$ is?

(d) Suppose it is known *a priori* that $H_0$ is true with probability $\pi_0$ and $H_1$ is true with probability $\pi_1 = 1 - \pi_0$. For what values of $\pi_0$ does the MAP decision rule always declare that $H_0$ is true, no matter what the value of the observation $X$ is?

**Problem 8** *(18 points, 6 points for each part)* The length $Y$ of a didjeridoo, a musical instrument coming from Austrailia, has the zipdingle distribution with expected value $E[Y] = 4$ and variance $Var(Y) = 25$. (The zipdingle distribution is very complicated.) A noisy measurement $X$ is available. It is assumed that $X = Y + W$ where the measurement noise $W$ is independent of $Y$ and $W$ is Gaussian with $E[W] = 0$ and $Var(W) = 10$.

(a) Find the estimator for $Y$ of the form $aX + b$ which minimizes the mean squared error over all such estimators.

(b) Find the estimator for $Y$ of the form $cX$ which minimizes the mean squared error over all such estimators.

(c) Which estimator has the smaller mean squared error, the best estimator of the form $aX + b$ or the best estimator of the form $cX$? Justify your answer.