University of Illinois at Urbana-Champaign

ECE 313: Probability with Engineering Applications

Fall 2002
Exam I

October 7, 2002

Name: _____

☐ Section C, 10:00 MWF ☐ Section D, 11:00 MWF

- You have **60 Minutes** for this exam.
- This exam is closed book; however, you may consult both sides of your $8.5'' \times 11''$ sheet of notes.
- Calculators, laptop computers, Palm Pilots, two-way e-mail pagers, etc. may not be used.
- Write your answers in the spaces provided.
- It is *not necessary* to convert answers that are simple common fractions into decimal fractions, but you should simplify the fractions by canceling common factors from the numerator and denominator (e.g., write 5/16 instead of 10/32).
- Note that the questions are not weighted equally. Budget your time accordingly and do not work too long on any one problem.
- **Please show all of your work. Answers without appropriate justification will receive very little credit.** If you need extra space, use the back of the previous page.

Score:

1. _____ (15 pts.)
2. _____ (20 pts.)
3. _____ (30 pts.)
4. _____ (15 pts.)

Total: _____(80 pts.)

Problem 1 *(15 points)* An urn contains balls numbered 1 through 6. Three balls are drawn randomly from the urn. Assume that all balls are equally likely to be drawn. Remember, answers without appropriate justification will receive very little credit.

Part (a) 5pts. What is the probability that ball 5 will be among the three balls that are drawn.

Probability ball 5 is drawn =

Part (b) 5pts. Once the three balls have been drawn, they are placed into a new urn. One ball is drawn at random from this urn. If that ball is the 5 ball, it is returned to the urn, and another ball is drawn. This is repeated until some ball other than the 5 is drawn (this might happen on the first try, or it might take several tries). Thus, the 5 ball cannot be chosen, regardless of whether or not it is in the new urn.
What is the probability that the chosen ball will have an even number?

Probability that the the chosen ball will have an even number =

Part (c) 5pts. Under the same conditions as for part (b), what is the probability that the 3 ball will be chosen?

Probability that the 3 ball will be chosen =

Problem 2 *(20 points)* Two fair dice are thrown. Let X_1 be the number shown on the first die and X_2 be the number shown on the second die. Define the random variable Z as follows:

$$Z = \begin{cases} 0 & : \quad X_2 \text{ is not evenly divisible by } X_1 \\ 1 & : \quad X_2 \text{ is evenly divisible by } X_1 \end{cases}$$

Note: X_2 is evenly divisible by X_1 if and only if X_2/X_1 is an integer.

Part (a) 5 pts. Compute the pmf for Z.

Part (b) 5 pts. Compute the conditional probability $P(\{Z = 1\} \mid \{X_2 = 4\})$.

$P(\{Z = 1\} \mid \{X_2 = 4\}) =$

Part (c) 5 pts. Compute the conditional probability $P(\{Z = 1\} \mid \{X_1 = 3\})$.

$P(\{Z = 1\} \mid \{X_1 = 3\}) =$

Part (d) 5 pts. Compute the conditional probability $P(\{X_1 = 3\} \mid \{Z = 1\})$.

$P(\{X_1 = 3\} \mid \{Z = 1\}) =$

Problem 3 *(30 points)* It is believed by many professors that a student's grade in ECE 359 can be predicted based on whether or not the student passed ECE 313. In fact, in the faculty handbook, one can find the following likelihood matrix:

Likelihood Matrix		Grade in ECE 359				
		A	B	C	D	F
$\mathbf{H}_0$:	Did not pass ECE 313	0	0	0.1	0.4	0.5
$\mathbf{H}_1$:	Passed ECE 313	0.4	0.3	0.2	0.1	0

A professor who is teaching ECE 359 for the first time is trying to determine whether a particular student passed ECE 313 based on the student's grade in ECE 359.

Part (a) 5 pts. Determine the the maximum likelihood decision rule and write it in the space below.

Part (b) 5 pts. While pondering the situation, the professor remembers that at the beginning of the semester 70% of the students in the class indicated that they had passed ECE 313. Thus, the professor determines that $\pi_1 = P(\mathbf{H}_1) = 0.7$. Using this information, fill in the joint probability matrix below.

Joint Prob. Matrix	Grade in ECE 359				
	A	B	C	D	F
$\mathbf{H}_0$					
$\mathbf{H}_1$					

Part (c) 5 pts. Assuming the same prior probabilities as in Part (b), what is the probability that a student obtains a passing grade in ECE 359?

P{ passing grade } =

Problem 3 *(cont)*

Part (d) 5 pts. Assuming the same prior probabilities as in Part (b), determine the the MAP decision rule and write it in the space below.

Part (e) 5 pts. Assuming the same prior probabilities as in Part (b), what is probability of error associated with the *maximum likelihood* decision rule?

P{ error } =

Part (f) 5 pts. Find the numbers π_{min} and π_{max} such that the ML and MAP decision rules are identical if and only if $\pi_{\min} \leq \pi_0 \leq \pi_{\max}$.

$\pi_{min} =$

$\pi_{max} =$

Problem 4 *(15 points)* A certain die is known to not be fair, but the probabilities associated with outcomes are not known. It *is* known, however, that the probabilities for rolling odd numbers are all equal, and that the probabilities for rolling even numbers are equal. In particular, we know the probabilities for two events:

$$\begin{aligned} P(\text{odd}) &= p \\ P(\text{even}) &= 1 - p \end{aligned} \tag{1}$$

Suppose that the die is rolled ten times, that the rolls are independent, and that the observed sequence is (odd,even,even,even,odd,even,even,even,even,even).

Part (a) 5 pts. Determine, in terms of p, the probability of this outcome.

Part (b) 5 pts. Determine, in terms of p, the probability of observing k odd rolls if the die is rolled n times.

Part (c) 5 pts. Given that the above sequence has been observed, determine the value of $\hat{p}_{ML}$, the Maximum Likelihood estimate for p. You must show all of your work to obtain any credit for this problem.