

Assigned: Wednesday, November 14, 2001

Due: Wednesday, November 28, 2001

Reminder: Happy Thanksgiving

Reading: Ross, Chapter 6 and Chapter 7

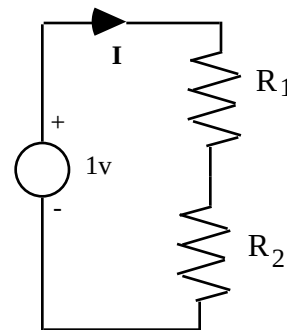
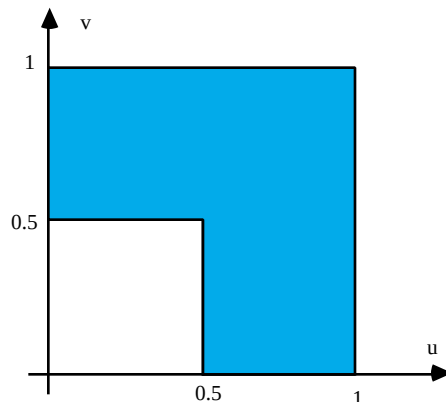
Noncredit Exercises: Ross, Chapter 6: Problems 26, 28-30, 41-43, 51, 54;

Theoretical Exercises: 8, 14, 22, 23, 33;

Chapter 7: Problems 1, 16, 26, 29, 34, 36; Theoretical Exercises: 1, 2, 17, 22, 23, 40

Problems:

1. Let (X, Y) have joint pdf $f_{X,Y}(u, v) = \begin{cases} C\sqrt{1-u^2-v^2}, & u^2+v^2 < 1, \\ 0, & \text{elsewhere.} \end{cases}$
 - (a) What is the value of C ?
 - (b) Find $P\{X^2+Y^2 < 0.25\}$.
2. The random point (X, Y) is uniformly distributed on the shaded region shown in the left-hand figure below.
 - (a) Find the marginal pdf $f_X(u)$ of the random variable X .
 - (b) Write down the marginal pdf $f_Y(v)$ of the random variable Y from your answer to part (b).
 - (c) Find $P\{X < Y < 2X\}$.
 - (d) What is $f_{X|Y}(u|\alpha)$, the conditional pdf of X given that $Y = \alpha$, if α satisfies $0 < \alpha < 1/2$?
What is $f_{X|Y}(u|\alpha)$, the conditional pdf of X given that $Y = \alpha$, if α satisfies $1/2 < \alpha < 1$?
Now, apply the theorem of total probability to compute the unconditional pdf of X from $f_{X|Y}(u|\alpha)$. Do you get the same answer as in part (a)?



3. Two resistors are connected in series to a one-volt voltage source as shown in the right-hand diagram above. Suppose that the resistance values R_1 and R_2 (measured in ohms) are independent random variables, each uniformly distributed on the interval $(0, 1)$. Find the pdf $f_I(a)$ of the current I (measured in amperes) in the circuit.
4. Let (X, Y) have joint pdf $f_{X,Y}(u, v) = \begin{cases} 2u, & 0 < u < 1, 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$
Find the pdf of $Z = X^2Y$.
5. (Unbelievable but true: this problem is easier than it looks...).
 - (a) If X is $N(0, \sigma^2)$, use the magic formula in Example 7b, Chapter 5.7 of Ross to show that X^2 has gamma pdf with parameter $(1/2, 1/2\sigma^2)$.
 - (b) Now, suppose that X , Y , and Z are independent $N(0, \sigma^2)$ random variables. Then X^2 , Y^2 , and Z^2 are independent gamma random variables with parameter $(1/2, 1/2\sigma^2)$. Use the

comment immediately following the proof of Proposition 3.1 (p. 267, 5th ed. or p. 262, 6th ed.) of Ross to *state* what the *type* of pdf of $\mathbf{W} = \mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2$ is, and write down *explicitly* the exact pdf. What is the numerical value of $f_{\mathbf{W}}(5)$ if $\sigma^2 = 4$?

- (c) Prove that $E[\mathbf{W}] = 3\sigma^2$. If you actually evaluated an integral to get this answer instead of using LOTUS, shame on you!
- (d) In a physical application, $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ represent the velocity (measured along three perpendicular axes) of a gas molecule of mass m . Thus, $\mathbf{H} = (1/2)m\mathbf{W}$ is the kinetic energy of the particle, and an important axiom of statistical mechanics asserts that the average kinetic energy is $E[\mathbf{H}] = E[(1/2)m\mathbf{W}] = (1/2)mE[\mathbf{W}] = (3/2)m\sigma^2 = (3/2)kT$ where k is Boltzmann's constant and T is the absolute temperature of the gas in $^\circ\text{K}$. (Note that the average energy is $(1/2)kT$ per dimension.) Show that the kinetic energy $\mathbf{H}$ has the Maxwell-Boltzmann pdf $f_{\mathbf{H}}(\beta) = \frac{2}{\sqrt{\pi}}(kT)^{-3/2}\sqrt{\beta}\exp(-\beta/kT)$, $\beta > 0$.
- (e) $V = \sqrt{\mathbf{W}} = \sqrt{\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2}$ is the "speed" of the molecule. Show that the pdf of V is $f_V(y) = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{3/2}y^2\exp\left(-\frac{my^2}{2kT}\right)$, $y > 0$ cf. Theoretical Exercise 1 of Chapter 5.
- (f) What is the average speed of the molecule?
6. The number of hours $\mathbf{R}$ that a student spends reading about probability in preparation for the ECE 313 Final Examination and the number of hours $\mathbf{S}$ that the student spends sleeping can be modeled as random variables with joint probability density function
- $$f_{\mathbf{R},\mathbf{S}}(x,y) = \begin{cases} K, & 10 \leq x + y \leq 20, x \geq 0, y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$
- (a) What is the value of K ?
- (b) What is the marginal pdf of $\mathbf{R}$?
- (c) Unfortunately, the more the student tries to read about probability, the more confused the student gets. Also, the less the student sleeps, the more tired the student gets. As a result, the student's *percentage* score $\mathbf{T}$ on the Final Exam is related to $\mathbf{S}$ and $\mathbf{R}$ via the equation $\mathbf{T} = 50 + 2.5(\mathbf{S} - \mathbf{R})$. Find the pdf of $\mathbf{T}$.
- (d) **Noncredit exercise:** Should $\mathbf{S}$ have denoted *s*tudying and $\mathbf{R}$ denoted *r*esting instead?