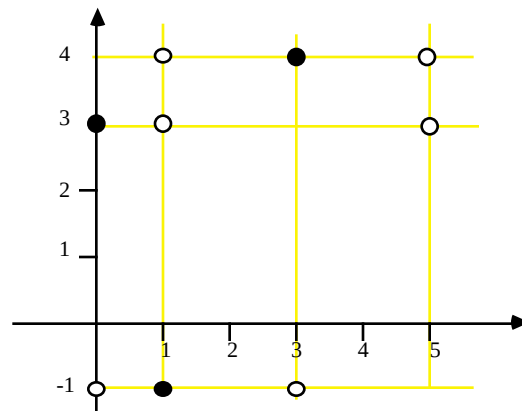


- 1.(a) In the figure below, I use $\circ$ to denote a probability mass of $1/12$ and $\bullet$ to denote a probability mass of $1/6$.



- (a) $p_X(0) = 1/6 + 1/12 = 1/4$, $p_X(1) = 1/3$, $p_X(3) = 1/4$, $p_X(5) = 1/6$.
 $p_Y(-1) = 2 \times 1/12 + 1/6 = 1/3$, $p_Y(3) = 1/3$, $p_Y(4) = 1/3$.
- (b) Since, for example, $p_X(3)p_Y(3) = (1/3) \times (1/4) = 1/12 \neq p_{X,Y}(3,3) = 0$, the random variables are not independent. We can also see this via the eyeball test: there are empty spots in the grid.
- (c) $P\{X \leq Y\} = p_{X,Y}(0,3) + p_{X,Y}(1,3) + p_{X,Y}(1,4) + p_{X,Y}(3,4) = 1/2$.
 $P\{X + Y \leq 8\} = 1 - p_{X,Y}(5,4) = 11/12$.
- (d) Given $Y = 3$, X takes on values 0, 1 and 5 with conditional probabilities $(1/6)/(1/3)$, $(1/12)/(1/3)$, and $(1/12)/(1/3)$, that is, $1/2$, $1/4$ and $1/4$ respectively. Hence, $E[X|Y=3] = 0 \cdot (1/2) + 1 \cdot (1/4) + 5 \cdot (1/4) = 3/2$, and $E[X^2|Y=3] = 0^2 \cdot (1/2) + 1^2 \cdot (1/4) + 5^2 \cdot (1/4) = 26/4$, giving $\text{var}(X|Y=3) = 26/4 - 9/4 = 17/4$.

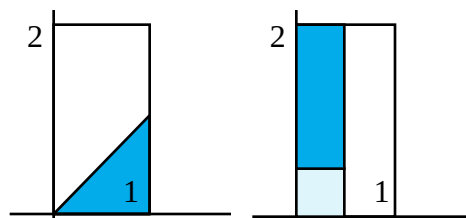
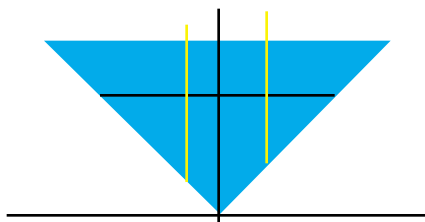
- 2.(a),(b) The joint pdf is nonzero on the quarter-plane shown. For $v > 0$, the pdf of Y is given by

$$f_Y(v) = \int_{u=-v}^v c \cdot (v^2 - u^2) \cdot \exp(-v) du = c \cdot [v^2 u - u^3/3]_{-v}^v \cdot \exp(-v) = c \cdot (4/3) \cdot v^3 \cdot \exp(-v). \text{ We recognize this as a } \gamma \text{ pdf with parameters } (4,1) \text{ which gives us that } c \cdot (4/3) = 1/\Gamma(4) = 1/3! \text{ and thus } c = 1/8.$$

$$\text{For } u > 0, f_X(u) = \int_{v=u}^{\infty} (1/8) \cdot (v^2 - u^2) \cdot \exp(-v) dv = (1/8) \cdot (-v^2 - 2v - 2 + u^2) \cdot \exp(-v) \Big|_u^{\infty} = (1/4) \cdot (1+u) \cdot \exp(-u)$$

while for $u < 0$, the integral is from $v = -u$ to ∞ . Thus, we get $f_X(u) = (1/4) \cdot (1+|u|) \cdot \exp(-|u|)$.

- (c) The pdf of X is symmetric and the integral of $u \cdot (1/4) \cdot (1+u) \cdot \exp(-u)$ from 0 to ∞ is finite (what's that got to do with it??) Hence, $E[X] = 0$.



- 3.(a) The computation of the double integral is left as an exercise since the final answer is known.

(b) $f_X(u) = \int_{v=0}^2 (6/7) \cdot (u^2 + uv/2) dv = (6/7) \cdot [u^2 v + uv^2/4]_{v=0}^2 = (6/7) \cdot (2u^2 + u)$ for $0 < u < 1$, and 0 otherwise.

(c) $P\{X > Y\} = P\{(X, Y) \text{ shaded triangle shown}\} = \int_{u=0}^1 \int_{v=0}^u (6/7) \cdot (u^2 + uv/2) dv du = 15/56$.

(d) $P\{Y > \frac{1}{2} \mid X < \frac{1}{2}\} = P\{Y > \frac{1}{2}, X < \frac{1}{2}\} / P\{X < \frac{1}{2}\}$
 $= P\{(X, Y) \text{ shaded rectangle shown}\} / P\{(X, Y) \text{ shaded rectangle+crosshatched square}\}$

$$\text{Now, } P\{X < \frac{1}{2}\} = \int_{u=0}^{1/2} (6/7) \cdot (2u^2 + u) du = (6/7) \cdot [2u^3/3 + u^2/2] \Big|_0^{1/2} = \frac{5}{28} \text{ while}$$

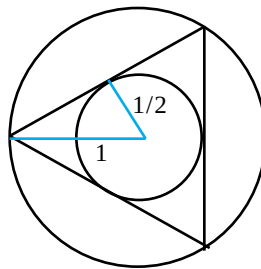
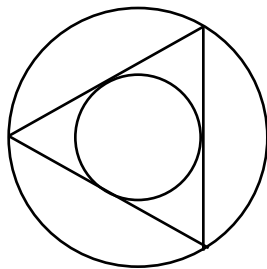
$$P\{Y > \frac{1}{2}, X < \frac{1}{2}\} = \int_{v=1/2}^2 \int_{u=0}^{1/2} (6/7) \cdot (u^2 + uv/2) du dv = \int_{v=1/2}^2 (6/7) \cdot [u^3/3 + u^2 v/4] \Big|_0^{1/2} dv$$

$$= \int_{v=1/2}^2 (3/28)(1/3 + v/2) dv = \frac{3}{28} \cdot \frac{23}{16}, \text{ giving that } P\{Y > \frac{1}{2} \mid X < \frac{1}{2}\} = \frac{69}{80}.$$

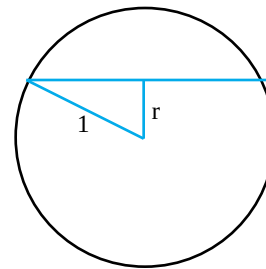
$$(e) \quad E[X] = \int_{u=0}^1 u \cdot (6/7) \cdot (2u^2 + u) du = (6/7) \cdot [u^4/2 + u^3/3] \Big|_0^1 = \frac{5}{7}$$

$$(f) \quad f_Y(v) = \int_{u=0}^1 (6/7) \cdot (u^2 + uv/2) du = (6/7) \cdot [u^3/3 + u^2 v/4] \Big|_0^1 = (6/7) \cdot (1/3 + v/4) \text{ for } 0 < v < 2, \text{ and } 0 \text{ otherwise.}$$

$$\text{Hence, } E[Y] = \int_{v=0}^2 v \cdot (6/7) \cdot (1/3 + v/4) dv = (6/7) \cdot [v^2/6 + v^3/12] \Big|_0^2 = \frac{8}{7}$$



$$\sin(\pi/6) = 1/2$$



- 4.(a) As is obvious from the left-hand figure, the chord is longer than the side of the inscribed equilateral triangle if its midpoint lies inside the inscribed circle. It is easily seen (or read in Ross!) that the radius of this inner circle is $1/2$. Hence, $P\{\text{random point lies inside inscribed circle}\} = \text{ratio of areas} = \pi(1/2)^2 / \pi(1)^2 = 1/4$. Thus, we can choose between $1/2$, $1/3$, and $1/4$ as the answer to the random chord problem!

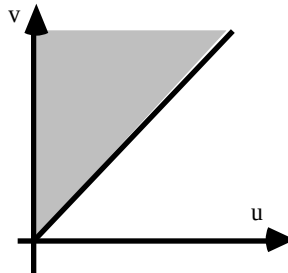
- (b) If the midpoint is at distance r from the center of the circle, the chord has length $2\sqrt{1 - r^2}$. Thus, we have that $L = 2\sqrt{1 - R^2} = 2\sqrt{1 - X^2 - Y^2}$ where R is the distance of the random point (X, Y) from the origin. As before, $0 \leq L \leq 2$, and for any x , $0 \leq x \leq 2$, we have that $F_L(x) = P\{L \leq x\}$

$$= P\{2\sqrt{1 - R^2} \leq x\} = P\{R \geq \sqrt{1 - x^2/4}\} = 1 - P\{R \leq \sqrt{1 - x^2/4}\} = 1 - \frac{\pi(1 - x^2/4)}{\pi 1^2} = x^2/4.$$

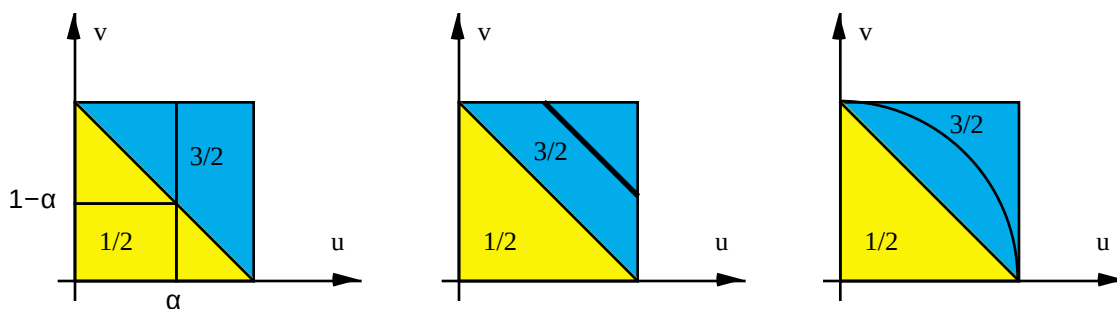
$$\text{Hence, } f_L(x) = \frac{d}{dx} F_L(x) = \begin{cases} x/2, & 0 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

- (c) $E[L] = \int_0^2 x \cdot x/2 dx = 4/3.$

- 5.(a) The pdf is nonzero in the shaded region shown.



- (b) $f_X(u) = \int_{v=u}^{v=\infty} 2 \exp(-u-v) dv = 2 \cdot \exp(-2u)$ if $u > 0$, and 0 otherwise.
 $f_Y(v) = \int_{u=0}^{u=v} 2 \exp(-u-v) du = 2 \cdot \exp(-v) - 2 \cdot \exp(-2v)$ if $v > 0$, and 0 otherwise.
- (c) The random variables are not independent: $f_X(u)f_Y(v) \neq f_{X,Y}(u,v)$.
- (d) $P\{Y > 3X\} = \int_{u=0}^{\infty} \int_{v=3u}^{\infty} 2 \exp(-u-v) dv du = \int_{u=0}^{\infty} 2 \exp(-4u) du = 1/2$.
- (e) $P\{X + Y < \alpha\} = \int_{u=0}^{\alpha/2} \int_{v=u}^{\alpha-u} 2e^{-u-v} dv du = \int_{u=0}^{\alpha/2} 2e^{-u} [e^{-u} - e^{-\alpha+u}] du = 1 - (1 + \alpha) \exp(-\alpha)$ for $\alpha > 0$.
 The probability is 0 for $\alpha \leq 0$.
- (f) $f_{X+Y}(\alpha) = \frac{d}{d\alpha} [1 - (1 + \alpha) \exp(-\alpha)] = \alpha \cdot \exp(-\alpha)$ for $\alpha > 0$, and 0 otherwise. This is a gamma density with parameters (2,1).



6. The joint pdf has values as shown in the two triangular regions.
- (a) $f_X(\alpha) = \text{area of cross-section at } \alpha = \text{sum of areas of two rectangles} = (1/2)(1 - \alpha) + (3/2)\alpha = (1/2) + \alpha$, for $0 \leq \alpha \leq 1$, and 0 if $\alpha < 0$ or $\alpha > 1$.
- (b) $P\{X + Y \leq 3/2\} = 1 - P\{X + Y > 3/2\} = 1 - \text{volume above heavy line in middle figure above}$
 $= 1 - (1/2) \times (1/2) \times (1/2) \times (3/2) = 13/16$.
- (c) $P\{X^2 + Y^2 \geq 1\} = \text{Volume outside heavy curve in right-hand figure above} = (1 - \pi/4) \times (3/2) = 3/2 - 3\pi/8$.
 (Alternatively, volume inside the curve is $(1/2)^2 + (\pi/4 - 1/2) \times 3/2 = 3\pi/8 - 1/2$)