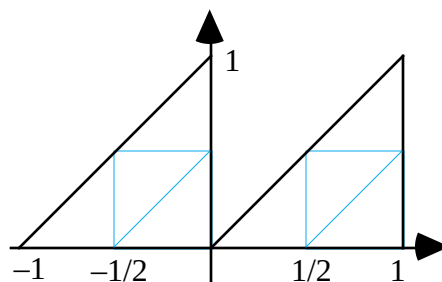


1. (a) This is a valid pdf. (b) This is a valid pdf.
(c) This is not a valid pdf because the function is negative for $0 < u < 1$.
However, $\int_0^1 \ln u \, du = u \ln u - u \Big|_0^1 = -1$, so $-\ln u, u \in (0,1)$ is a valid pdf.
(d) This is not a valid pdf because the function is negative for $u \in (0,1)$. Also, since the function is positive for $u \in (1,2)$, $Cf(u)$ cannot be a valid pdf for any C . (e) This is a valid pdf.
(f) Although $\int_0^3 (2/3)(u-1) \, du = (1/3)(u-1)^2 \Big|_0^3 = 1$, this is not a valid pdf because the function is negative for $u \in (0,1)$. Also, since the function is positive for $u \in (1,3)$, $Cf(u)$ cannot be a valid pdf for any C .
(g) Since $\int_0^\infty \exp(-2u) \, du = 1/2$, this is not a valid pdf. However, $2 \exp(-2u), u \in (0, \infty)$ is a valid pdf; it is the exponential density with parameter 2.
(h) Although $\int_0^\infty 4\exp(-2u) - \exp(-u) \, du = 2 - 1 = 1$, this is not a valid pdf because the function is negative for $u > \ln 4$. Since the function is positive for $u < \ln 4$, $Cf(u)$ cannot be a valid pdf for any C .
2. The pdf is as shown on the diagram below where some lines have been added to aid in computation. Each triangle shown has area $1/8$.



- (a) By inspection, we see that $P\{|X| < 1/2\} = 1/2$. Similarly, $P\{X < 1/2\} = 5/8$ and $P\{X > 0 \cap \{X < 1/2\}\} = P\{0 < X < 1/2\} = 1/8$, giving $P\{X > 0 \mid X < 1/2\} = 1/5$. Politically correct anti-segregationists (i.e. those who believe in integration) who failed to sketch the pdf can proceed as follows:
$$P\{|X| < 1/2\} = \int_{-1/2}^0 (1+u) \cdot du + \int_0^{1/2} u \cdot du = \int_{-1/2}^0 1 \cdot du + \int_{-1/2}^{1/2} u \cdot du = u \Big|_{-1/2}^0 + \frac{u^2}{2} \Big|_{-1/2}^{1/2} = \frac{1}{2} + 0 = \frac{1}{2} \text{ and so on.}$$

(b)
$$E[X] = \int_{-\infty}^{\infty} u \cdot f_X(u) \cdot du = \int_{-1}^0 u \cdot (1+u) \cdot du + \int_0^1 u \cdot u \cdot du = \frac{u^2}{2} + \frac{u^3}{3} \Big|_{-1}^0 + \frac{u^3}{3} \Big|_0^1 = \frac{1}{6}.$$

(c)
$$E[|X|] = \int_{-\infty}^{\infty} |u| \cdot f_X(u) \cdot du = \int_{-1}^0 -u \cdot (1+u) \cdot du + \int_0^1 u \cdot u \cdot du = \frac{-u^2}{2} + \frac{-u^3}{3} \Big|_{-1}^0 + \frac{u^3}{3} \Big|_0^1 = \frac{1}{2}.$$
3. (a) No, the tank is empty after 500 gallons have been sold, and towards the end of the week, the people asking for the extra 180 gallons are s.o.l. (b) Sure, with 70 gallons to spare.
(c) The weekly demand can be satisfied if it (the demand) does not exceed 0.5. Thus,
$$P\{X \leq 0.5\} = \int_0^{0.5} 5(1-u)^4 \, du = -(1-u)^5 \Big|_0^{0.5} = 1 - \left(\frac{1}{2}\right)^5 = \frac{31}{32}$$

(d) We want the smallest value of C such that $P\{X > C\} \leq 10^{-5}$. But,
$$P\{X > C\} = \int_C^1 5(1-u)^4 \, du = -(1-u)^5 \Big|_C^1 = (1-C)^5 \leq 10^{-5} \text{ if } C \geq 0.9. \text{ Thus, a 900 gallon tank is required to achieve the desired goal.}$$

(e),(f),(g)

$Y = \begin{cases} X & \text{if } X \leq C, \\ C & \text{if } X > C. \end{cases}$ Hence, the weekly gross profit is $640Y$, and the **average** weekly **gross** profit is

$$E[640Y] = \int_0^C 5(1-u)^4 640u \, du + \int_C^1 5(1-u)^4 640C \, du = \frac{640}{6}[1 - (1-C)^6].$$
 As a function of C , the size of the

tank, this increases from 0 if $C = 0$ (no tank!) to $\$640/6$ if $C = 1$. Now, the average **net** profit $= E[640Y - 20C] = E[640Y] - 20C = (640/6)[1 - (1-C)^6] - 20C$. This has a maximum if C satisfies $640(1-C)^5 = 20$ i.e. $(1-C)^5 = (1/32)$, i.e. if $C = 1/2$. Thus, a 500 gallon tank that costs $\$10$ per week to rent gives an average gross profit of $\$630/6$ and the maximum average net profit of $\$570/6$ per week. In contrast, a full-sized 1000 gallon tank gives only a slightly larger gross profit of $\$640/6$ per week, but a smaller net profit of $\$520/6$ per week. Of course, there may be intangible losses in goodwill if the tank is emptied before the week ends. However, the probability of this happening is only $1/32$ when $C = 1/2$.

$$4.(a) \quad E[Y] = E[X^2] = \int_{-1}^{+1} u^2(1/2)du = 1/3. \quad E[Y^2] = E[X^4] = \int_{-1}^{+1} u^4(1/2)du = 1/5.$$

$$\text{Hence, } \text{var}(Y) = E[Y^2] - (E[Y])^2 = (1/5) - (1/3)^2 = 4/45.$$

$$(b) \quad E[Z] = \int_{-\infty}^{\infty} g(u)f(u)du = \int_{-1}^0 -u^2(1/2)du + \int_0^1 u^2(1/2)du = -1/3 + 1/3 = 0.$$

5.(a) Since the pdf is nonzero only in the interval (a, b) , we have that $E[X] = \int_a^b u \cdot f(u) \, du$. But, the integrand satisfies $a \cdot f(u) < u \cdot f(u) < b \cdot f(u)$ for all u such that $a < u < b$. The comparison property for integrals then shows that (a) holds.

(b) The function $v = (u-\mu)^2$ on (a, b) is bounded above by the straight line through the points $(a, (a-\mu)^2)$ and $(b, (b-\mu)^2)$, which has equation $v = \mu^2 - ab + [(b+a) - 2\mu]u$.

$$\text{Thus, } \text{var}(X) = \int_a^b (u-\mu)^2 \cdot f(u) \, du < \int_a^b (\mu^2 - ab + [(b+a) - 2\mu]u) \cdot f(u) \, du = \mu^2 - ab + (b+a-2\mu) \cdot \mu = -(\mu-a)(\mu-b).$$

Note that the variance must be positive since part(a) shows that $a < \mu < b$ and thus the factors $(\mu-a)$ and $(\mu-b)$ have opposite signs. It is easily shown via calculus that $-(\mu-a)(\mu-b)$, regarded as a function of μ , has a maximum value of $(b-a)^2/4$ at $\mu = (a+b)/2$. Hence, $\text{var}(X) < (b-a)^2/4$. As a point of interest, the uniform distribution on (a, b) has variance $(b-a)^2/12$, smaller by a factor of 3.

(c) Yes, the results hold for discrete random variables as well. The same argument applies with sums replacing the integrals.

(d) The discrete random variable Y taking on values a and b each with probability $1/2$ has $\mu = (a+b)/2$ and variance $(b-a)^2/4$ since all the mass is at distance $(b-a)/2$ from the mean. If the random variable values are restricted to (a, b) , then the variance is strictly smaller than $(b-a)^2/4$. However, with both discrete and continuous random variables, we can approach the maximum variance value as closely as we like, i.e. for any $\epsilon > 0$, no matter how small, we can achieve a variance of $(b-a)^2/4 - \epsilon$.