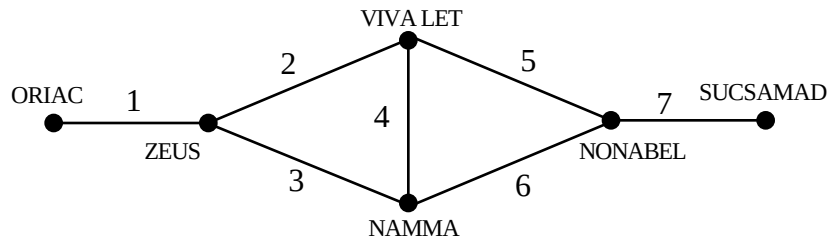


- 1.(a) With N-ary replication of each component, each replicated component has probability of failure p^N since all the N copies of the component must fail for the replicated component to fail. The probability that all M replicated components are in working order is thus $(1 - p^N)^M$, and the probability of failure for the system in the left-hand figure is thus $1 - (1 - p^N)^M$. On the other hand, suppose that the entire system is replicated N times as shown in the right-hand figure. Since each copy of the entire system fails with probability $1 - (1-p)^M$, the probability of failure for the system in the right-hand figure is $[1 - (1-p)^M]^N$.
- (b) $1 - (1 - 0.2^N)^5 \leq 0.001 \quad (1 - 0.2^N)^5 \geq 0.999 \quad (1 - 0.2^N) \geq 0.99979992 \quad 0.2^N \leq 0.000200080$
 $N \geq 6$. More generally, $N \geq \frac{\ln(1 - (1-\alpha)^{1/M})}{\ln p}$ where α is the desired maximum failure probability.
- $[1 - (1 - 0.2)^5]^N \leq 0.001 \quad 0.67232^N \leq 0.001 \quad N \geq 18$. More generally, $N \geq \frac{\ln \alpha}{\ln(1 - (1-p)^M)}$
- Put another way, in our car example, having a single gigantic car with 6 engines, 6 transmissions, 6 brakes, ... will provide more reliable transportation than having 17 separate cars. This may (or may not!) explain why people buy one large SUV rather than 18 compact cars!
- (c) If $M = 1000$, we get $N \geq 9$ in the one case. The other case blows up on many calculators because the quantity $1 - (1-p)^M \approx 1 - \exp(-Mp)$ (huh?) $= 1 - \exp(-200)$ is very nearly 1. The Unix high-precision utility *bc* gives its value as 0.999...99861610344 where there are 87 9's preceding the 8616.... Since most cars have 1000 (or more!) parts, the number of separate cars required to provide reliable transportation is far larger than the number of cars in this world! On the other hand, a car with nine engines, nine transmissions etc will provide the necessary reliability but it will be difficult to squeeze into most parking spaces!
- Moral:** Replicate at the component level, instead of replicating entire systems. It's a lot cheaper!

2. Linguistic innocents are reminded that both Arabic and Hebrew are read from right to left. Number the links as shown. Let V_i denote the event that the i-th link is viable (i.e. working) and let A denote the event that it is possible to call from ORIAC to SUCSAMAD.



- (a) If V_4^c has occurred, i.e. the VIVA LET to NAMMA link has been severed, then,
 $P(A|V_4^c) = P(V_1 V_2 V_5 V_7 \mid V_1 V_3 V_6 V_7 | V_4^c) = P(V_1 V_2 V_5 V_7 \mid V_1 V_3 V_6 V_7)$ by independence!
 $= P(V_1 V_2 V_5 V_7) + P(V_1 V_3 V_6 V_7) - P(V_1 V_2 V_3 V_5 V_6 V_7) = 2(1-p)^4 - (1-p)^6$ by independence! On the other hand, if V_4 has occurred, i.e the VIVA LET to NAMMA link is viable, then
 $P(A|V_4) = P(V_1(V_2 \mid V_3)(V_5 \mid V_6)V_7|V_4) = P(V_1(V_2 \mid V_3)(V_5 \mid V_6)V_7)$ by independence!
 $= P(V_1) \times P(V_2 \mid V_3) \times P(V_5 \mid V_6) \times P(V_7) = (1-p)^2 \times [(1-p) + (1-p) - (1-p)^2]^2 = (1-p)^2(1-p^2)^2$
 $= (1-p)^4(1+p)^2$ by repeated use of the independence of events. Hence,
 $P(A) = p \times [2(1-p)^4 - (1-p)^6] + (1-p) \times (1-p)^4(1+p)^2 = (1-p)^4 \times (1 + 2p + p^2 - 2p^3)$
- (b) $P(V_3^c | A) = P(A \cap V_3^c) / P(A)$. But, $P(A \cap V_3^c) = P(V_1 V_2 (V_5 \mid V_4 V_6) V_7)$
 $= (1-p)^2 \times [(1-p) + (1-p)^2 - (1-p)^3] \times (1-p) = (1-p)^4 \times (1 + p - p^2)$. Hence,
 $P(V_3 | A) = 1 - P(V_3^c | A) = 1 - \frac{(1-p)^4 \times (1 + p - p^2)}{(1-p)^4 \times (1 + 2p + p^2 - 2p^3)} = \frac{p(1 + 2p - 2p^2)}{1 + 2p + p^2 - 2p^3}$.
- (c) Clearly X is nonnegative, and cannot exceed 70. Also, $X = 0$ if either link #1 or link #7 has been severed i.e. if $V_1^c \mid V_7^c$ occurs. Thus, assume that $V_1 V_7$ has occurred. As in part(a), we can calculate the capacity given that V_4^c has occurred and also given that V_4 has occurred. Thus, given that $V_1 V_4^c V_7$ has occurred, the capacity from ORIAC to SUCSAMAD is 20 if only one of the paths from ZEUS to NONABEL is working, and 40 if both are working, etc.. These capacities are indicated on the two "Karnaugh" maps shown on the next page.

		V₁V₄V₇ occurred			
V ₂ V ₅ →		00	01	11	10
V ₃ V ₆ ↓	00	0	0	20	0
	01	0	0	20	0
	11	20	20	40	20
	10	0	0	20	0

		V₁V₄V₇ occurred			
V ₂ V ₃ →		00	01	11	10
V ₅ V ₆ ↓	00	0	0	0	0
	01	0	20	20	20
	11	0	50	70	20
	10	0	50	50	20

From this, we get $P\{X = 70\} = (1-p)^7$ {**check:** all links must work to get a capacity of 70},
 $P\{X = 50\} = [(1-p)^2 - (1-p)^4](1-p)^3 = (1-p)^5 - (1-p)^7 = (1-p)^5[2p - p^2]$ {**check:** links 1, 3, 4, 5, 7 must work, and at least one of 2 and 6 must have failed which has prob $p + p - p^2$, **why?**},
 $P\{X = 40\} = (1-p)^6 p$ {**check:** all links except 4 must be in working condition},
 $P\{X = 20\} = 2(1-p)^4 p^3 + 7(1-p)^5 p^2 + 2(1-p)^6 p$, {**check:** the first term corresponds to 1-2-5-7 and 1-3-6-7 as the only working links, the last term corresponds to 3 or 5 being the only failed link, and the middle term corresponds to everything working **except** (2 and exactly one of 4 and 5) or (3 and exactly one of 4, 5, 6) or (4 and exactly one of 5 and 6)},
 $P\{X = 0\} = 1 - P\{X \neq 0\} = 1 - [(1-p)^7 + (1-p)^5 - (1-p)^7 + p(1-p)^6 + 2(1-p)^4 p^3 + 7(1-p)^5 p^2 + 2(1-p)^6 p]$
 $= 1 - (1-p)^4(1 + 2p + p^2 - 2p^3) = 1 - P(A)$ as it should be!
 Finally, $E[X] = 70P\{X = 70\} + 50P\{X = 50\} + 40P\{X = 40\} + 20P\{X = 20\}$
 $= 20(1-p)^7 + 80(1-p)^6 p + 50(1-p)^5 + 40(1-p)^4 p^3 + 140(1-p)^5 p^2$. As a quick check, note that $E[X] = 70$ if $p = 0$ (totally incompetent terrorists!) and $E[X] = 0$ if $p = 1$ (fearless liberation fighters not afraid to die in the attempt!). It is interesting to plot these probabilities and expected values as a function of p . $P(A) \approx 80\%$ for $p = 0.05$ and decreases to 50% at roughly $p = 0.2419...$ when $E[X] = 21....$

3.(a) $P\{X = 4\} = P\{A \text{ wins 4 straight}\} + P\{B \text{ wins four straight}\} = (1/2)^4 + (1/2)^4 = 1/8$.

$$P\{X = 5\} = 2 \cdot P\{A \text{ loses only one games \#1-\#4}\} = 2 \cdot \binom{4}{1} \cdot (1/2)^5 = 1/4$$

$$P\{X = 6\} = 2 \cdot P\{A \text{ loses 2 games out of games \#1-\#5}\} = 2 \cdot \binom{5}{2} \cdot (1/2)^6 = 5/16$$

$$P\{X = 7\} = 2 \cdot P\{A \text{ loses 3 games out of games \#1-\#6}\} = 2 \cdot \binom{6}{3} \cdot (1/2)^7 = 5/16 \text{ also.}$$

$E[X] = 4 \cdot (1/8) + 5 \cdot (1/4) + (6 + 7) \cdot (5/16) = 5.8125$ games. Both teams are equally likely to win the Series, and both are equally likely to win in five games.

- (b) Let p denote the chances of winning at home and $q = 1-p$ the chances of winning on the road.
 In both formats, the first four games are two at home and two on the road. Hence, A and B have the same chance $p^2 q^2$ of winning the Series in four games. Thus, $P\{X = 4\} = 2p^2 q^2$. If a team is leading 3-1 after four games, then either it has lost one of its two home games or one of its two road games, and the probability of this is $[2 \cdot p^3 q + 2 \cdot q^3 p] = 2pq(p^2 + q^2)$ for each team. Hence, the Series is tied 2-2 after 4 games with probability $1 - 2p^2 q^2 - 4pq(p^2 + q^2)$. Alternatively, the Series is tied 2-2 after 4 games with probability $p^4 + q^4 + 4p^2 q^2$ where the terms can be identified as the chances of home team always wins, home team always loses, and one loss in each stadium. Exercise: prove the two expressions are equal.
 Reality check: $p^4 + q^4 + 4p^2 q^2 + 2p^2 q^2 + 2 \cdot [2 \cdot p^3 q + 2 \cdot q^3 p] = (p + q)^4 = 1$ by the binomial theorem.
 Returning to the case when one team leads 3-1 after 4 games, A wins the Series (at home) in 5 games with probability $2pq \cdot (p^2 + q^2) \cdot p$ in the AABBBAA format and in 5 games (on the road) with probability $2pq \cdot (p^2 + q^2) \cdot q$ in the AABBBAA format. B wins in 5 on the road with probability $2pq \cdot (p^2 + q^2) \cdot q$ in the AABBBAA format, and at home with probability $2pq \cdot (p^2 + q^2) \cdot p$ in the AABBBAA format. Thus,
 $P\{X = 5\} = 2pq \cdot (p^2 + q^2)$ in either format.
 Next, consider that if A leads 3-1 after 4 games, then it wins the Series 4-2 (on the road) with probability $2pq \cdot (p^2 + q^2) \cdot q \cdot q$ in the AABBBAA format and (at home) with probability $2pq \cdot (p^2 + q^2) \cdot p \cdot p$ in the AABBBAA format. If B leads 3-1, then it wins the Series 4-2 (at home) with probability $2pq \cdot (p^2 + q^2) \cdot p \cdot p$ in the AABBBAA format and (on the road) with probability $2pq \cdot (p^2 + q^2) \cdot q \cdot q$ in the AABBBAA format.
 On the other hand, if the Series is tied 2-2 after 4 games, then in either format, the next two games are home and away for both teams. Thus, A and B both have equal chance $p \cdot q$ of winning both. Hence,
 A wins 4-2 with probability $2pq \cdot (p^2 + q^2) \cdot q \cdot q + pq \cdot (p^4 + q^4 + 4p^2 q^2) = 6p^3 q^3 + 3pq^5 + p^5 q$ in the AABBBAA format (the terms correspond to one home and one road loss, two home losses, two road losses respectively). A wins 4-2 with probability $2pq \cdot (p^2 + q^2) \cdot p \cdot p + pq \cdot (p^4 + q^4 + 4p^2 q^2) = 6p^3 q^3 + pq^5 + 3p^5 q$ in the AABBBAA format. For B, the probabilities of winning 4-2 are interchanged for the two formats.

Thus, adding up the probabilities, we get $P\{X = 6\} = 12p^3q^3 + 4pq^5 + 4p^5q$ in either format. Finally, the Series is tied 3-3 with probability $p^6 + q^6 + 9p^4q^2 + 9p^2q^4$ where the terms correspond to each team winning all its home games, each team losing all its home games, each team wins only twice at home, and each home team wins only once at home. Hence, we get that $P\{X = 7\} = p^6 + q^6 + 9p^4q^2 + 9p^2q^4$ in either format. Furthermore, the probability that A wins in 7 games is just $p \cdot P\{X = 7\}$ in either format while B wins in 7 with probability $q \cdot P\{X = 7\}$. Finally,

$P\{A \text{ wins the Series}\} = P\{A \text{ in } 4\} + P\{A \text{ in } 5\} + P\{A \text{ in } 6\} + P\{A \text{ in } 7\}$
 $= p^2q^2 + [2pq \cdot (p^2 + q^2) \cdot p] + [6p^3q^3 + 3pq^5 + p^5q] + p \cdot P\{X = 7\}$ in the AABBBABA format while
 $P\{A \text{ wins the Series}\} = p^2q^2 + [2pq \cdot (p^2 + q^2) \cdot q] + [6p^3q^3 + pq^5 + 3p^5q] + p \cdot P\{X = 7\}$ in the AABBBBAA format. A miracle of modern mathematical calculation shows that these two expressions are the same, i.e. neither format is better for A. The incredulous are invited to try their hand at the algebra!

Numerically, for $p = 0.55$, X takes on values 4, 5, 6, and 7 with probabilities 0.1225125, 0.249975, 0.3131184375, and 0.3143940625 respectively for either format, and $E[X] = 5.8193940625$. Similarly, $P\{A \text{ wins Series}\} = 0.515719703125$.

- (c) The home field advantage for A over B is relatively slight; 0.515719703125 as compared to 0.5 chance of winning the Series. Neither format is better. But B has a slightly better chance of winning in 4 or 5 in the AABBBBAA format while A has the better chance of winning in 4 or 5 in the AABBBABA format (which can be readily explained by considering the question: who has played more home games?)
- (d) Given that the Series ended in 5 games, A has *a posteriori* probability p of winning the Series in the AABBBABA format and *a posteriori* probability q of winning the Series in the AABBBBAA format.

- 4.(a) The pmf of X is Poisson with parameter λp . Hence, $P(A_n) = P\{X = n\} = \exp(-\lambda) \cdot \lambda^n / n!$.

- (b) Given that the event $A_n = \{X = n\}$ has occurred, Y is binomial(n, p) and hence, $E[Y | A_n] = np$. Thus, the

unconditional mean is $E[Y] = \sum_{n=0}^{\infty} E[Y | A_n] \cdot P(A_n) = \sum_{n=0}^{\infty} (np) \cdot \exp(-\lambda) \cdot \lambda^n / n! = p \cdot E[X] = p \cdot \lambda$.

- (c) If $Y = k$, then there must have been *at least* k α -particles emitted by the source, i.e. we know that X must

have been k or more. Hence, $P\{Y = k\} = \sum_{n=k}^{\infty} P\{Y = k\} | A_n \cdot P(A_n) = \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot \exp(-\lambda) \cdot \lambda^n / n!$

$$= \frac{(\lambda p)^k}{k!} \exp(-\lambda) \sum_{n=k}^{\infty} (1-p)^{n-k} \cdot \lambda^{n-k} / (n-k)! = \frac{(\lambda p)^k}{k!} \exp(-\lambda) \cdot \exp(\lambda(1-p)) = \frac{(\lambda p)^k}{k!} \exp(-\lambda p) = P(B_k).$$

We conclude that the *unconditional* pmf of Y is Poisson with parameter λp .

- (d) Conditioned on $B_k = \{Y = k\}$, the random variable X can take on values k or more only. In fact, it follows from Bayes' theorem that for any $n \geq k$,

$$P\{X = n | B_k\} = \frac{P\{B_k | X = n\} \cdot P\{X = n\}}{P(B_k)} = \frac{\binom{n}{k} p^k (1-p)^{n-k} \cdot \exp(-\lambda) \cdot \frac{\lambda^n}{n!}}{\frac{(\lambda p)^k}{k!} \exp(-\lambda p)} = \exp[-\lambda(1-p)] \cdot \frac{[\lambda(1-p)]^{n-k}}{(n-k)!}$$

This is essentially a Poisson pmf shifted right by k , that is, the conditional pmf of X is the pmf of the random variable $k + Z$ where Z is a Poisson random variable with parameter $\lambda(1-p)$.

- (e),(f) $E[X | B_k] = E[k + Z] = k + \lambda(1-p)$. Since $P(B_k)$ was found in (c), we have that $E[X] = \sum_{k=0}^{\infty} E[X | B_k] \cdot P(B_k)$

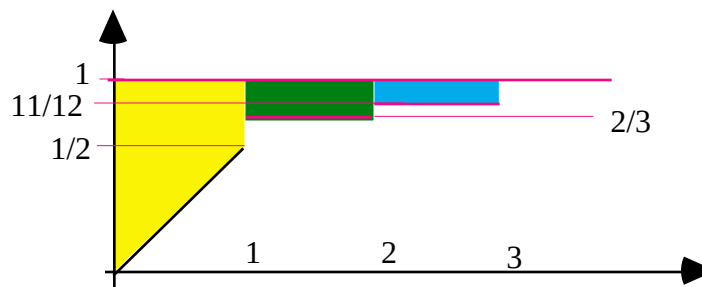
$$= \sum_{k=0}^{\infty} (k + \lambda(1-p)) \cdot \exp(-\lambda p) \cdot (\lambda p)^k / k! = \sum_{k=0}^{\infty} k \cdot \exp(-\lambda p) \cdot (\lambda p)^k / k! + \lambda(1-p) \sum_{k=0}^{\infty} \exp(-\lambda p) \cdot (\lambda p)^k / k! = \lambda.$$

- (g) What value of n maximizes $P\{Y = k\}$? We solved a similar problem in Problem 3 of Problem Set #4. The *ratio* of the probabilities of $\{Y = k\}$ under two assumptions about n is

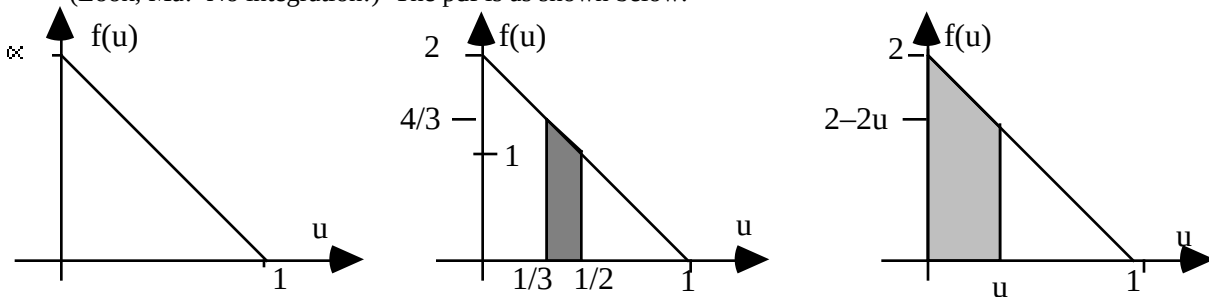
$$\frac{f(n)}{f(n-1)} = \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n-1}{k} p^k (1-p)^{n-1-k}} = \frac{n(1-p)}{n-k} > 1 \text{ if } np < k \text{ and } < 1 \text{ if } np = k. \text{ Hence, the largest probability of}$$

the observation $\{Y = k\}$ occurs when n is the largest integer such that $np \leq k$, i.e. $n = \lceil k/p \rceil$

5. Since X takes on positive values only, $E[X] = \int_0^{\infty} P\{X > u\} du = \text{area between CDF and line at height 1 to the right of the origin} = (1 - (1/2) \cdot (1/2) \cdot 1) + 1/3 + 1/12 = 7/6$ (see the three areas shaded below).



6. (Look, Ma! No integration!) The pdf is as shown below.



The total area under the curve is $(1/2) \times \alpha \times 1 = \alpha/2$ which gives $\alpha = 2$.

- (a) $P\{6X^2 > 5X - 1\} = P\{6X^2 - 5X + 1 > 0\} = P\{(3X - 1)(2X - 1) > 0\}$
 $= P[\{X > 1/3\} \cap \{X > 1/2\}] \cup \{X < 1/3\} \cap \{X < 1/2\}] = P\{X > 1/2\} \cup \{X < 1/3\}$
 $= 1 - P[1/3 \leq X \leq 1/2]$. Now, $P[1/3 \leq X \leq 1/2] = \text{area under the pdf from } 1/3 \text{ to } 1/2$.
 This is the difference of two triangular areas and is thus $(1/2) \times [(4/3) \times (2/3) - 1 \times (1/2)] = 7/36$.
 Hence, $P\{6X^2 > 5X - 1\} = 1 - P[1/3 \leq X \leq 1/2] = 1 - 7/36 = 29/36$.
- (b) $F_X(u)$ is the area under the pdf upto the point u . By inspection, we have that $F_X(u) = 0$ if $u < 0$ and $F_X(u) = 1$ if $u > 1$. For $0 \leq u \leq 1$, $F_X(u)$ is the area shaded in the right-hand figure above
 $= 1 - (1/2) \times (1-u) \times (2-2u) = 1 - (1-u)^2 = 2u - u^2$.
- (a) $\exp(x)$ is an increasing function, and $\exp(0) = 1$. Since $\lambda > 0$, $\exp(\lambda(u-n/2)) > 1$ for $u > n/2$. The series of inequalities is now obvious. In the sum for $P\{X > n/2\}$, we replace $1 \cdot p_X(u)$ by $\exp(\lambda(u-n/2)) \cdot p_X(u)$ to get an upper bound, and then expand the sum to all $u \geq 0$ to get an even looser bound. Finally, we invoke LOTUS in reverse to write the upper bound as an expectation.
- (b) LOTUS gives $E[\exp(\lambda X)] = \sum_{i=0}^n \exp(\lambda i) \cdot \binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=0}^n \binom{n}{i} (\exp(\lambda) \cdot p)^i (1-p)^{n-i} = (p \cdot \exp(\lambda) + 1 - p)^n$
 by the binomial theorem. Substituting in the Chernoff bound, we get $P\{X > n/2\} \leq E[\exp(\lambda(X-n/2))]$
 $= \exp(-\lambda n/2) \cdot E[\exp(\lambda X)] = \exp(-\lambda n/2) \cdot (1 - p + p \cdot \exp(\lambda))^n$ for $\lambda > 0$.
- (c) The bound has value $2^{-n/2} \cdot (1+p)^n$ for $\lambda = \ln 2$ and $p^{n/2} \cdot (2-p)^n \approx (4p)^{n/2}$ if p is small and $\lambda = -\ln p$.
- (d) The derivative is $(-n/2) \cdot \exp(-\lambda n/2) \cdot (1-p+p \cdot \exp(\lambda))^n + \exp(-\lambda n/2) \cdot n \cdot (1-p+p \cdot \exp(\lambda))^{n-1} \cdot p \cdot \exp(\lambda)$ and this equals 0 if $(-n/2) \cdot (1-p+p \cdot \exp(\lambda)) + n \cdot p \cdot \exp(\lambda) = 0$, i.e. at $\exp(\lambda) = (1-p)/p$ which gives a minimum.
 The value of the bound at $\exp(\lambda) = (1-p)/p$ is $[p/(1-p)]^{n/2} \cdot (2 \cdot (1-p))^n = (2\sqrt{p(1-p)})^n = (4p)^{n/2} \cdot (1-p)^{n/2}$.
 For small values of p , $(1-p)$ is nearly 1, $\ln(1-p)$ is nearly 0 and thus
 $\lambda = \ln((1-p)/p) = \ln(1-p) - \ln p \approx -\ln p$ where the bound is has slightly larger value $(4p)^{n/2}$.
- (e) If $p = 0.01$, $(2\sqrt{p(1-p)})^{n/2} \leq 10^{-5}$ for $n \geq 8$ which is a *huge* improvement over parts (c) or (d).
 Who gets the raise? The balding guy with pointy tufts of hair, of course!