1.(a) From the theorem of total probability, $p_n = (1/3)p_{n-1} + (2/3)q_{n-1} = (2/3) - (1/3)p_{n-1}$ since $p_{n-1} + q_{n-1} = 1$. Similarly, $q_n = (2/3)p_{n-1} + (1/3)q_{n-1} = (2/3) - (1/3)q_{n-1}$. The solutions to these difference equations are easily derived by substituting $p_n = a + b\alpha^n = (2/3) - (1/3)p_{n-1} = (2/3) - (1/3)(a + b\alpha^{n-1})$, which simplifies to $b\alpha^{n-1}(\alpha + (1/3)) + ((4/3)a - (2/3)) = 0$. Since this must hold *for all values of n,* it must be that $a = (1/2)$ and $\alpha = (-1/3)$. Thus, $p_n = a + b\alpha^n = (1/2) + b(-1/3)^n$ for all n. Since $p_1 = 2/3$, we get $b = 1/2$, and thus $p_n = (1/2)[1 - (-1/3)^n]$. Similarly, $q_n = a + b\alpha^n = (1/2) + b(-1/3)^n$ for all n. But, since $q_1 = 1/3$, we get $b = -1/2$, and hence $q_n = (1/2)[1 + (-1/3)^n]$.

(b) Yes, the game is asymptotically fair since both p_n and q_n approach 1/2 in the limit as $n \to \infty$.

(c) Wilma wins if the outcomes are HH, TT, HTHH, THTT, HTHTHH, THTHTT, ... with probability (1/2) + (1/8) + (1/32) + … = (1/2)×(1/[1–1/4]) = 2/3 while Fred wins with probability 1/3. Thus, $q_1 = 2/3$ and $p_1 = 1/3$. The rest of the analysis is exactly as in parts (a)-(b) with Wilma and Fred having exchanged their roles. In essence, the person tossing first in a game is merely setting up whether in this particular game an H wins or a T wins, and the second person tossing then is "going first" in the game of part (a).

2.(a) $\text{LR} = \dfrac{P_1(\text{observation})}{P_0(\text{observation})} = \dfrac{(1-p_1)^{33}p_1}{(1-p_0)^{33}p_0}$. More generally, if the k-th pin is heard, the LR is $\dfrac{(1-p_1)^{k-1}p_1}{(1-p_0)^{k-1}p_0}$.

(b) The log likelihood ratio is $(k-1)\times\ln\left(\dfrac{1-p_1}{1-p_0}\right) + \ln\left(\dfrac{p_1}{p_0}\right)$. Since $1 - p_1 < 1 - p_0$, the coefficient of (k–1) is negative, and after cross-multiplying, the maximum-likelihood decision rule can be stated as:

Choose Sprint if $\mathbf{X} < \theta_{ML}$ and choose AT&T if $\mathbf{X} > \theta_{ML}$ where $\theta_{ML} = 1 + \dfrac{\ln\left(\dfrac{p_0}{p_1}\right)}{\ln\left(\dfrac{1-p_1}{1-p_0}\right)}$.

Reality check: Since it is twice as likely that we hear the pin drop on Sprint than on AT&T, hearing it relatively soon should favor Sprint over AT&T, and the decision rule does just that.

(c) For the given values of p_0 and p_1, $\theta_{ML} = 34.62\ldots$ so the maximum-likelihood decision rule chooses Sprint. The actual value of LR is 1.01… for k = 34 and .99… for k = 35. Although not asked for, we can easily find the Type I error (false alarm) probability = P(decide Sprint when AT&T is actually the carrier) = P{first pin drop heard on trial #1 or trial #2 or … or trial #34) = 1 – P{heard nothing on first 34 trials} = $1 - (1 - p_0)^{34} = 0.49686\ldots$ Similarly, the Type II error (missed detection or miSprint) probability = P{decide AT&T when Sprint is actually the carrier} = P{heard nothing on first 34 trials} = $(1 - p_1)^{34}$ = 0.24958…

(d) Now we have to compare LR to $\pi_0/\pi_1 = 19$, or equivalently, the log likelihood ratio to ln 19 = 2.9444… The Bayesian decision rule is to choose Sprint if $\mathbf{X} < \theta_{Bayes}$ and choose AT&T if $\mathbf{X} > \theta_{Bayes}$ where

$$\theta_{Bayes} = 1 + \frac{\ln\left(\dfrac{\pi_0}{\pi_1}\right) + \ln\left(\dfrac{p_0}{p_1}\right)}{\ln\left(\dfrac{1-p_1}{1-p_0}\right)} = 1 + \frac{\ln\left(\dfrac{p_0\pi_0}{p_1\pi_1}\right)}{\ln\left(\dfrac{1-p_1}{1-p_0}\right)}$$

Since the numerator is positive and the denominator is negative, $\theta_{Bayes} < 1$, and thus the Bayesian decision is to always choose AT&T. We could have obtained this even more easily by observing that LR has value 2 if k = 1, and that since $1 - p_1 < 1 - p_0$, LR decreases as k increases. Thus, LR is always smaller than $\pi_0/\pi_1 = 19$, and hence the Bayesian decision is to always choose AT&T. The false alarm probability is 0 while the missed detection probability is 1. Hence, the average error probability is 0×0.95 + 1×0.05 = 0.05. In contrast, the maximum-likelihood decision rule has average error probability 0.49686×0.95 + 0.24958×0.05 = 0.48449… One can do almost as well by just tossing Miss Bergen's dime and deciding AT&T or Sprint according as it comes up Heads or Tails!

(e) You mean you can't tell?

3.(a,b) If H_0 is the true hypothesis, $P\{\mathbf{X} = 86\} = \binom{90}{86}(0.9)^{86}(0.1)^4$ while $P\{\mathbf{X} = 96\} = 0$.

If H_1 is the true hypothesis, $P\{\mathbf{X} = 86\} = \binom{90}{71}(0.9)^{71}(0.1)^{19}$ while $P\{\mathbf{X} = 96\} = \binom{90}{81}(0.9)^{81}(0.1)^9$.

The likelihood ratio has value $\frac{86\times85\times84\times\ldots\times72}{5\times6\times7\ldots\times19}\times\left(\frac{1}{9}\right)^{15} = 0.027$ if $\mathbf{X} = 86$, and value ∞ if $\mathbf{X} = 96$.

Hence, if $\mathbf{X} = 86$, the maximum-likelihood decision is that the connecting flight is late, while if $\mathbf{X} = 96$, the maximum-likelihood decision is that the connecting flight is on time. Indeed, for any $\mathbf{X} > 90$, the decision must be that the connecting flight is on time, as should be obvious from first principles.

(c) The Bayesian decision rule compares the likelihood ratio to $\pi_0/\pi_1 = 2$, and hence, in both cases, the Bayesian decision is the same as the maximum-likelihood decision.

(d) Let C denote the events that all passengers who show up get seats. Then, $P(C|H_0) = 1$, while $P(C|H_1) = P\{\mathbf{Y} \le 85\} = 1 - P\{\mathbf{Y} > 85\} = 1 - [P\{\mathbf{Y} = 90\} + P\{\mathbf{Y} = 89\} + P\{\mathbf{Y} = 88\} + P\{\mathbf{Y} = 87\} + P\{\mathbf{Y} = 86\}] = 0.953...$. Hence, $P(C) = P(C|H_0)\pi_0 + P(C|H_1)\pi_1 = 1\times(2/3) + (0.953...)\times(1/3) = 0.984...$ and $P(H_0|C) = P(C|H_0)\pi_0/P(C) = (2/3)/P(C) = 0.688...$

4.(a) The bill passes if the Conservative Republicans and at least one other group votes for it, i.e. the event $A\cap(B\ \ C\ \ D)$ occurs, or if all groups except the Conservative Republicans vote for it, i.e. the event $A^c\cap B\cap C\cap D$ occurs. Note that these are disjoint events (Why?). Now, $(B\ \ C\ \ D)^c = B^c\cap C^c\cap D^c$ by DeMorgan's theorem, and thus we can partition A into two disjoint events as follows:
$A = A\cap(B\ \ C\ \ D)\quad A\cap B^c\cap C^c\cap D^c$. Hence, we see that
$P(A\cap(B\ \ C\ \ D)) = P(A) - P(A\cap B^c\cap C^c\cap D^c) = P(A) - P(A)P(B^c)P(C^c)(D^c)$ by independence. Thus,
$P(\text{bill passes}) = P(A\cap(B\ \ C\ \ D)) + P(A^c\cap B\cap C\cap D) = P(A) - P(A)P(B^c)P(C^c)P(D^c) + P(A^c)P(B)P(C)P(D)$ (independence again!) $= 0.9\times[1 - 0.4\times0.5\times0.8] + 0.1\times0.6\times0.5\times 0.2 = 0.756 + 0.006 = 0.762$.

(b) The Conservative Republicans must be in favor for the motion to override to pass. Furthermore, since politics makes strange bedfellows, either the Liberal Democrats must also be in favor or both the other groups must also be in favor. Hence,
$P(\text{motion to override passes}) = P[E(H\quad FG)] = P(E)\times P(H\quad FG)$ by independence of E and $H\quad FG$)
$= P(E)\times[P(H)+P(FG) - P(FGH)] = P(E)\times[P(H) + P(F)P(G) - P(F)P(G)P(H)]$ (by independence again!)
$= 0.99\times[0.1 + 0.4\times0.6 - 0.6\times0.4\times0.1] = 0.31284$.

5.(a) The game winner wins the last point in the game, i.e., a score of 4-x or x-4 (with $0 \le x < 3$) means that the x points won by the loser must have been among the first 3+x points. Thus,
$P(4\text{-}x) = \binom{3+x}{x} p^3q^xp$, $P(x\text{-}4) = \binom{3+x}{x} q^3p^xq$, giving $P(4\text{-}0) = p^4$; $P(4\text{-}1) = 4p^4q$; $P(4\text{-}2) = 10p^4q^2$;
$P(0\text{-}4) = q^4$; $P(1\text{-}4) = 4pq^4$; $P(2\text{-}4) = 10p^2q^4$.

(b) Deuce is reached with probability $\binom{6}{3}p^3q^3 = 20p^3q^3$. To show that the sum is 1, just multiply it all out and hope you don't make an error (or use Mathematica).

(c) $P(\text{A wins next two points}|\text{deuce}) = p^2$. $P(\text{B wins next two points}|\text{deuce}) = q^2$.
$P(\text{each wins one point}|\text{deuce}) = P(\text{deuce again}) = 2pq$ (why 2 ?)

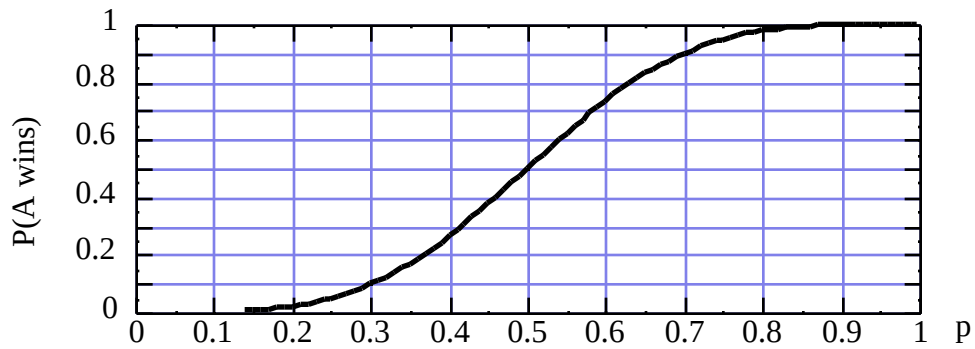
(d) $P(\text{A wins the game}|\text{deuce}) = p^2 + (2pq)p^2 + (2pq)^2p^2 + \ldots = p^2/(1 - 2pq)$. Similarly,
$P(\text{B wins the game}|\text{deuce}) = q^2 + (2pq)q^2 + (2pq)^2q^2 + \ldots = q^2/(1\text{-}2pq)$. As a sanity check, we note that $1 = (p+q)^2 = p^2 + q^2 + 2pq$ so that $1 - 2pq = p^2+q^2$ implying that $p^2/(1\text{-}2pq) + q^2/(1\text{-}2pq) = 1$, i.e. one of A and B *does* ultimately win the game.

(e) $P(\text{A wins}) = f(p) = p^4 + 4p^4q + 10p^4q^2 + \frac{20p^5q^3}{1 - 2pq} = p^4 + 4p^4q + \frac{10p^4q^2}{1 - 2pq}$. Clearly,
$f(0) = 0$ (can't win games without winning points); $f(1) = 1$ (can't lose if you win every point) and $f(0.5) = 0.5$ (if A and B are equally likely to win each point, any argument purporting to show that A wins games more often than B can be turned around to show that B wins more often than A!) Finally, note that f(q) is just the win probability for B and hence $f(p) + f(q) = 1$.

(f) $f'(p) = 4p^3 + 16p^3q - 4p^4 + \frac{(1 - 2pq)(40p^3q^2 - 20p^4q) - (-2q+2p)10p^4q^2}{(1-2pq)^2}$ and $f'(0.5) = 2.5$ so that $f(0.5+\varepsilon) = f(0.5) + \varepsilon\bullet f'(0.5) + ... = 0.5 + 2.5\varepsilon + ...$ Thus, for players of *almost* equal skill, the change in the probability of winning a game is 2.5 times the change in the probability of winning a point (i.e. the rules of tennis amplify any small deviations from 0.5 in the *point-win* probability to give 2.5 times as large deviations from 0.5 in the *game-win* probability. Such amplification is much stronger for sets and

matches, cf. "The Drunken Tennis Player" in Ian Stewart's *Game, Set and Math*, Penguin Books, 1991.

(g) f(p) looks vaguely like an elongated S. $f(p) \geq (2/3)$ for $p \geq 0.5685...$



6. Let $A_2, A_3, \ldots, A_{12}$ denote the events that 2, 3, 4, . . . , 12 is rolled with two fair dice.

(a) P(winning on first roll) = $P(A_7 \quad A_{11}) = P(A_7) + P(A_{11}) = 6/36 + 2/36 = 2/9$.
P(losing on the first roll) = $P(A_2 \quad A_3 \quad A_{12}) = 1/36 + 2/36 + 1/36 = 1/9$.

(b) P(shooter's point is 4) = $P(A_4) = 3/36 = 1/12 = P(A_{10})$ = P(shooter's point is 10).
P(shooter's point is 5) = $P(A_5) = 4/36 = 1/9 = P(A_9)$ = P(shooter's point is 9).
P(shooter's point is 6) = $P(A_6) = 5/36 = P(A_8)$ = P(shooter's point is 8).

(c) For disjoint events A and B, P(event A occurs before B) = P(A)/(P(A) + P(B)). Hence,
P(shooter makes point|shooter's point is 4) = 3/(3+6) = 1/3 = P(shooter makes point|shooter's point is 10)
P(shooter makes point|shooter's point is 5) = 4/(4+6) = 2/5 = P(shooter makes point|shooter's point is 9)
P(shooter makes point|shooter's point is 6) = 5/(5+6) = 5/11 = P(shooter makes point|shooter's point is 8)

(d) P(winning at craps) = P(winning on first roll)+ $\sum_i$P(making point|point is i)P(shooter's point is i)
$= (8/36) + 2\times[(3/36)\times(1/3) + (4/36)\times(2/5) + (5/36)\times(5/11)] = (8/36) + (2/36)\times[1 + 8/5 + 25/11]$
$= (8/36) + (2/36)\times[55 + 88 + 125]/55 = (440 + 536)/(36\times55) = 244/495 = 0.49292929\ldots$

(e) P(making 8 the hard way) = P(rolling 4-4 before rolling 7 or rolling 8 the easy way)
$= (1/36)/(1/36 + (6/36 + 4/36)) = 1/11$. Thus, in a long sequence of trials when the shooter's point is 8, one-eleventh of the time you win \$10 while ten-eleventh of the time you lose your \$1. Thus, in the long run, you neither make nor lose money on the average, and the bet is perfectly fair.