

1. Let $f^{(k)}(x)$ denote the k -th derivative of $f(x)$. Note that $f^{(0)}(x) = f(x)$. Assume that all the derivatives exist.

(a) $f(x) = f^{(0)}(0) + \frac{f^{(1)}(0)}{1!} \cdot x + \frac{f^{(2)}(0)}{2!} \cdot x^2 + \dots + \frac{f^{(k)}(0)}{k!} \cdot x^k + \dots$

(b) If $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$, then $f^{(n)}(x) = n! a_n = \text{constant}$ and all higher order derivatives are zero.

(c) True.

(d)-(f) $\frac{d}{dx}(1+x)^n = f^{(1)}(x) = n \cdot (1+x)^{n-1}$. $\frac{d^2}{dx^2}(1+x)^n = f^{(2)}(x) = n(n-1) \cdot (1+x)^{n-2}$, and more generally, $f^{(k)}(x) =$

$n(n-1)(n-2)\dots(n-k+1) \cdot (1+x)^{n-k}$ for $1 \leq k \leq n$, while $f^{(k)}(x) = 0$ for $k > n$. Note that $f^{(n)}(x) = n!$

Hence, $f(x) = (1+x)^n = 1 + \frac{n}{1!} \cdot x + \frac{n(n-1)}{2!} \cdot x^2 + \dots + \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \cdot x^k + \dots + \frac{n!}{n!} \cdot x^n$

$$= \binom{n}{0} + \binom{n}{1} \cdot x + \binom{n}{2} \cdot x^2 + \dots + \binom{n}{k} \cdot x^k + \dots + \binom{n}{n} \cdot x^n$$

(g) False. (h)-(i) $\frac{d}{dx}(1-x)^n = f^{(1)}(x) = (-n) \cdot (1-x)^{n-1}(-1) = n \cdot (1-x)^{n-1}$.

$\frac{d^2}{dx^2}(1-x)^n = f^{(2)}(x) = n(n-1) \cdot (1-x)^{n-2}(-1) = n(n-1) \cdot (1-x)^{n-2}$, and more generally,

$f^{(k)}(x) = n(n-1)(n-2)\dots(n-k+1) \cdot (1-x)^{n-k}$ for all $k > 0$. Notice the difference between this and part (f).

Hence, the Taylor series for $(1-x)^n$ is $1 + \frac{n}{1!} \cdot x + \frac{n(n-1)}{2!} \cdot x^2 + \dots + \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \cdot x^k + \dots$ and

does not terminate at degree n . The $(n+1)$ -th term is $\frac{n(n+1)(n+2)\dots(2n)}{(n+1)!} \cdot x^{n+1} = \frac{(n+2)\dots(2n)}{(n-1)!} \cdot x^{n+1}$

(note the cancellation) while, the $(n+i)$ -th term is $\frac{n(n+1)\dots(2n+i-1)}{(n+i)!} \cdot x^{n+i} = \frac{(n+i+1)\dots(2n+i-1)}{(n-1)!} \cdot x^{n+i}$ where

the numerator and denominator are each the product of $n-1$ consecutive integers.

(j) Setting $n = 1$ and 2 respectively in the above results, we get

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots \quad \text{and} \quad \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

(k) When the Taylor series has infinitely many terms, we need to worry about convergence. For any given n , the absolute value of the ratio of the $(n+i+1)$ -th term to the $(n+i)$ -th term is $\frac{2n+i}{n+i} |x|$ which converges to $|x|$ as $i \rightarrow \infty$. The ratio test then says that the Taylor series for $(1-x)^n$ converges if $|x| < 1$.

2.(a) If $f(x) = (1-x)^n$, then for $1 \leq k \leq n$, $f^{(k)}(x) = n(n-1)(n-2)\dots(n-k+1) \cdot (1-x)^{n-k}(-1)^k$. Hence

$$(1-x)^n = 1 - \frac{n}{1!} \cdot x + \frac{n(n-1)}{2!} \cdot x^2 + \dots + (-1)^k \cdot \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} \cdot x^k + \dots + (-1)^n \cdot \frac{n!}{n!} \cdot x^n$$

$$= \binom{n}{0} - \binom{n}{1} \cdot x + \binom{n}{2} \cdot x^2 + \dots + (-1)^k \binom{n}{k} \cdot x^k + \dots + (-1)^n \binom{n}{n} \cdot x^n \text{ while}$$

$$(1+x)^n = \binom{n}{0} + \binom{n}{1} \cdot x + \binom{n}{2} \cdot x^2 + \dots + \binom{n}{k} \cdot x^k + \dots + \binom{n}{n} \cdot x^n$$

(b) The absolute values of the coefficients are the same in both expansions, but the even powers of x have positive signs in both expansions while the odd powers of x have opposite signs in the two expansions.

(c) The odd powers of x cancel out when we add $(1+x)^n$ and $(1-x)^n$

(d) $(1+1)^n + (1-1)^n = 2^n = 2 \cdot \left[\binom{n}{0} + \binom{n}{2} + \dots \right] = 2 \cdot [\text{total number of subsets of size } k, k \text{ even}]$

(e) Since there are a total of 2^n subsets, and 2^{n-1} of them contain even numbers of elements, the rest must contain odd numbers of elements, no? Else, consider $h(x) = (1+x)^n - (1-x)^n$ and observe that

$$h(1) = 2^n = 2 \cdot \left[\binom{n}{1} + \binom{n}{3} + \dots \right] = 2 \cdot [\text{total number of subsets of size } k, k \text{ odd}]$$

3. E, F, and G are events defined on a sample space

(a) Only E occurs: EF^cG^c . Note that the expression E means that E occurs without excluding F or G; they may or may not have occurred. (b) Both E and G but not F: EF^cG .

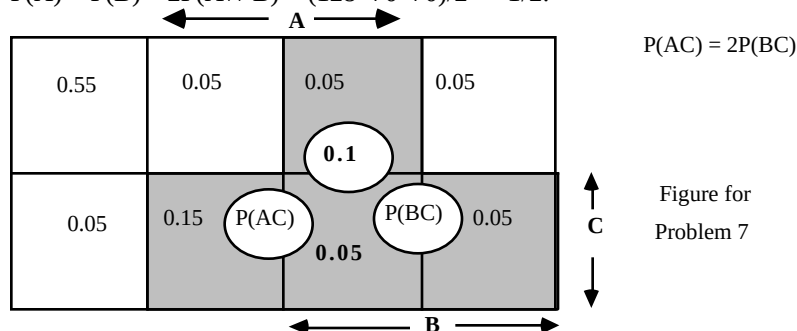
(c) At least one of the three occur. $E \cup F \cup G$. The more cumbersome expression $E \cup F \cup G \cup EF \cup EG \cup FG \cup EFG$ is equivalent to the simpler one (Prove it!)

(d) At least two occur. $EF \cup EG \cup FG$. (e) All three occur. EFG . (f) None occur. $E^cF^cG^c$

(g) At most one occurs. $E^cF^c \cup E^cG^c \cup F^cG^c$ (Compare (d)). The given expression is just the one describing the event that at least two of the events E^c , F^c , G^c occurred (and hence at most one of E, F, G occurred!)

(h) At most two occur. $E^c \cup F^c \cup G^c$ (Compare (c)). This is just the event that at least one of the events E^c , F^c , G^c occurred (i.e. at most two of E, F, G occurred). Alternatively, $(EFG)^c$ since all three do not occur.

- (i) Exactly two occur. $EFG^c \quad EF^cG \quad E^cFG$ or $(EF \quad EG \quad FG) - EFG$. (j) At most three occur. Ω
- 4.(a) Two men can be chosen from the six in $\binom{6}{2} = 15$ ways and one woman can be chosen in $\binom{6}{1} = 6$ ways giving a total of 90 different {2M,1W}. Interchanging man and woman in the above statement shows that there are 90 {2W,1M} committees. Since there are $\binom{12}{3} = 220$ committees that can be chosen, each of these committees occurs with probability $9/22$. We can cross-check our work by recalling that there are $\binom{6}{3} = 20$ {3M} committees and 20 {3W} committees. $90+90+20+20 = 220$, and we have counted them all.
- (b) Since each specialty flavor must contain at least one essence, the question is asking for the number of non-empty subsets of a set of 5 essences. The answer is $2^5 - 1 = 31$.
- 5.(a) $A^c \leftrightarrow \bar{a}$, $A \cap B \leftrightarrow a \cdot b$ and $A \cap B \leftrightarrow a \cdot b$ where all operations are performed bit-by-bit on the words.
- (b) If $A \leftrightarrow a = (a_1, a_2, \dots, a_n)$, then $P(A) = \sum_{i=1}^n p_i a_i = \mathbf{p} \cdot \mathbf{a}$ for those who have had Math 315...
- (c) $P(A) + P(A^c) = 1$ for any event A. Thus, we consider all the events in pairs $\{A, A^c\}$ and note that there are 2^{n-1} such pairs (one pair is $\{\Omega, \emptyset\}$), with each pair contributing 1 to the sum of the probabilities of all the events. Hence, the sum of the probabilities of all the events is 2^{n-1} , and the average probability is $1/2$. An alternative solution is as follows. $\sum P(A) = \sum \mathbf{p} \cdot \mathbf{a} = \mathbf{p} \cdot (\sum \mathbf{a}) = \mathbf{p} \cdot (2^{n-1}, 2^{n-1}, \dots, 2^{n-1}) = 2^{n-1}$ because arranging all the $\mathbf{a}$'s into a $2^n \times n$ table just gives us an n-variable truth table in which we know that each column has 2^{n-1} ones and 2^{n-1} zeroes. The average probability is thus $1/2$.
- (d) If the outcome is x_3 , then the 2^{n-1} events which have occurred correspond to those n-bit words $\mathbf{a}$ with a 1 in the third position. The remaining $n-1$ bits take on all possible values (i.e., form a $(n-1)$ -variable truth-table. Hence, $\sum P(A)$ over all the events that have occurred $= 2^{n-1} p_3 + 2^{n-2}(p_1 + p_2 + p_4 + p_5 + \dots + p_n) = 2^{n-1} p_3 + 2^{n-2}(1 - p_3) = 2^{n-2}(1 + p_3)$. Dividing by 2^{n-1} , we conclude that the average probability of the events that have occurred is $1/2 + p_3/2$. Similarly, the events which did not occur have average probability $1/2 - p_3/2$.
- 6.(a) $A = \{\text{ddddddd1: d = don't care if it is 0 or 1}\}$ giving $|A| = 2^7 = 128$; $P(A) = 2^7/2^8 = 1/2$.
- (b) The shift register contains 4 1's which can be chosen from the 8 positions in $\binom{8}{4} = \frac{8 \times 7 \times 6 \times 5}{1 \times 2 \times 3 \times 4} = 70$ ways (the remaining 4 positions have 0's in them.) Hence, B contains 70 outcomes and $P(B) = 70/2^8 = 35/128$.
- (c) AB is the set of shift register contents with 4 1's, one of which is in the least insignificant bit position. The remaining 3 1's can be in any of $\binom{7}{3} = \frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35$ positions. Hence, $P(A \cap B) = 35/2^8$;
- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = (128+70-35)/2^8 = 163/2^8$;
- $P(A \cup B) = P(A) + P(B) - 2P(A \cap B) = (128+70-70)/2^8 = 1/2$.



7. We use the Venn diagram/Karnaugh map shown above.
- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.4$, and hence $P(A^c B^c) = 1 - P(A \cup B) = 0.6$
- $P(AB \cup AC \cup BC) = 0.3 = P(AB) + P(AC) + P(BC) - 2P(ABC) = 0.1 + 3 \times P(BC) - 2 \times 0.05$. Hence, $P(BC) = 0.1$, $P(AC) = 0.2$, $P(AC \cup BC) = 0.2 + 0.1 - 0.05 = 0.25$. Thus, $P(A^c B^c C) = P(\text{snaps and crackles only}) = P(C) - P(AC \cup BC) = 0.05$, and $P(A^c B^c C^c) = P(\text{Rice Krispies}) = P(A^c B^c) - P(A^c B^c C) = 0.55$.
- $P(AB^c C^c) = P(A) - P(AB \cup AC) = 0.2 - 0.1 - 0.1 + 0.05 = 0.05$.
- Similarly, $P(A^c B C^c) = P(B) - P(AB \cup BC) = 0.3 - 0.1 - 0.2 + 0.05 = 0.05$.