1.(a) Let $\mathbf{X}$ denote a continuous random variable whose pdf $f_{\mathbf{X}}(u)$ is an *even function* of u, i.e. $f_{\mathbf{X}}(u) = f_{\mathbf{X}}(-u)$. Let $F_{\mathbf{X}}(u)$ denote the CDF of $\mathbf{X}$, and let $\alpha > 0$.

TRUE	FALSE	
□	■	$F_{\mathbf{X}}(\alpha) = F_{\mathbf{X}}(-\alpha)$
■	□	$P\{\mathbf{X} > \alpha\} = F_{\mathbf{X}}(-\alpha)$
■	□	$P\{\div\mathbf{X}\div < \alpha\} = 2F_{\mathbf{X}}(\alpha) - 1$
□	■	$P\{\div\mathbf{X}\div > \alpha\} = 2F_{\mathbf{X}}(\alpha)$
■	□	$P\{\div\mathbf{X}\div > \alpha\} = 2[1 - F_{\mathbf{X}}(\alpha)]$

(b) $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ denote continuous random variables with finite variance. Suppose that $\mathbf{Y} = -\mathbf{X}$, and $\mathbf{Z} = 2\mathbf{X}$.

TRUE	FALSE	
■	□	$\rho_{\mathbf{X},\mathbf{Y}} = -1$
□	■	$\rho_{\mathbf{X},\mathbf{Z}} = +2$
■	□	$f_{\mathbf{Y}}(u) = f_{\mathbf{X}}(-u)$
■	□	$f_{\mathbf{Z}}(u) = (1/2)f_{\mathbf{X}}(u/2)$
□	■	Since $\mathbf{Z} = \mathbf{X} + \mathbf{X}$, $f_{\mathbf{Z}} = f_{\mathbf{X}} * f_{\mathbf{X}}$
□	■	$F_{\mathbf{Y}}(u) = F_{\mathbf{X}}(-u)$
□	■	$F_{\mathbf{Z}}(u) = (1/2)F_{\mathbf{X}}(u/2)$

2.(a) Given n random variables $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$, $E[\mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_n] = E[\mathbf{X}_1] + E[\mathbf{X}_2] + \ldots + E[\mathbf{X}_n]$

- □ **only if** the random variables are *independent.*
- □ **only if** the random variables are *uncorrelated.*
- □ **only if** the random variables are *jointly Gaussian.*
- □ **only if** the random variables have *zero means.*
- ■ **always**.

(b) Given n random variables $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$, $\mathrm{var}[\mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_n] = \mathrm{var}[\mathbf{X}_1] + \mathrm{var}[\mathbf{X}_2] + \ldots + \mathrm{var}[\mathbf{X}_n]$

- □ **if** the random variables have *identical marginal distributions.*
- ■ **if** the random variables are *uncorrelated.*
- □ **if** the random variables are *jointly Gaussian.*
- □ **if** the random variables have *zero means.*
- □ **always**.

3.(a) Three boxes, of course!

(b) $P\{\mathbf{X} > 3\} = 1 - P\{$first three boxes contain LDJ, DJL, JLD, LJD, JDL, DLJ$\} = 1 - 6/27 = 7/9$.

(c) Let us condition on the first box containing a picture of Luke. Then, $P\{\mathbf{X} > n \mid \text{first box is L}\}$

$= P[\{$n–1 succeeding boxes contain only L or D$\}$ $\{$n–1 succeeding boxes contain only L or J$\}$
$= (2/3)^{n-1} + (2/3)^{n-1} - (1/3)^{n-1} = (2^n - 1)/3^{n-1} = P\{\mathbf{X} > n \mid \text{first box is D}\} = P\{\mathbf{X} > n \mid \text{first box is J}\}$.
Hence, $P\{\mathbf{X} > n\} = (2^n - 1)/3^{n-1}$ for $n \ge 3$. Note that this gives 7/9 for $P\{\mathbf{X} > 3\}$, as it should.
Alternatively, $P\{\mathbf{X} > n\} = \{$first n boxes have at most two different types of pictures$\}$

$= P[\{$n boxes have L and/or D only$\}$ $\{$n boxes have L and/or J only$\}$ $\{$n boxes have D and/or J only$\}$
$= (2/3)^n + (2/3)^n + (2/3)^n - (1/3)^n - (1/3)^n - (1/3)^n = (2^n - 1)/3^{n-1}$ since the events {LL…L}, {DD…D}, and {JJ…J} of probability $(1/3)^n$ have been double counted in $(2/3)^n + (2/3)^n + (2/3)^n$.

(d) For $n < 3$, $P\{\mathbf{X} > n\} = 1$, of course.

(e) $$E[\mathbf{X}] = \sum_{n=0}^{\infty} P\{\mathbf{X} > n\} = 1 + 1 + 1 + \sum_{n=3}^{\infty} \frac{2^n - 1}{3^{n-1}} = 3 + 3\bullet\left(\frac{2}{3}\right)^3 \bullet \frac{1}{1-2/3} - \left(\frac{1}{3}\right)^2 \bullet \frac{1}{1-1/3} = \frac{11}{2} = 5\frac{1}{2}.$$

An alternative method of solution is to argue as follows. The first box gives Jimmy a picture that he did not have before. Then, the succeeding boxes contain the same picture with probability 1/3, and a different picture with probability 2/3. Hence, the average number of boxes to get the second picture is 3/2. Finally, having got two pictures, the third picture is in succeeding boxes with probability 1/3, and hence 3 boxes are required on average. Thus, an average of $1 + 3/2 + 3 = 5.5$ boxes are needed, just as shown above.

4.(a) $\mathbf{X}$ is an exponential random variable with parameter λT,
or $\mathbf{X}$ is a gamma random variable with parameters $(1, \lambda T)$. In either case, $P\{0 < \mathbf{X} \le T\} = 1 - \exp(-\lambda T)$

(b) $P(A) = P\{$exactly one arrival in $(0, T]\} = P\{$Poisson RV with parameter λT has value $1\} = \lambda T \bullet \exp(-\lambda T)$.

(c) The answers are different because the events are different. If $\mathbf{X} \le T$, it does not necessarily mean that there was exactly one arrival in (0, T]. However, if there was exactly one arrival in (0, T], then we know for sure that $\mathbf{X} \le T$. Thus, the event A is a subset of the event $\{\mathbf{X} \le T\}$.

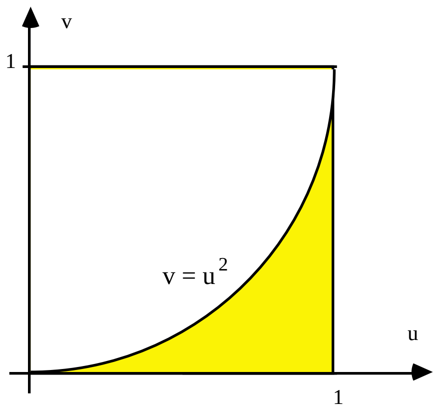
(d) $P\{\mathbf{X} \le \tau \mid A\} = P[\{\mathbf{X} \le \tau\} \cap A]/P(A) = P[\{1 \text{ arrival in } (0, \tau]\} \cap \{\text{no arrivals in } (\tau, T]\}]/P(A)$
$= P\{1 \text{ arrival in } (0, \tau]\} \bullet P\{\text{no arrivals in } (\tau, T]\}/P(A)$ by independence of arrivals in disjoint intervals
$= [\lambda\tau \bullet \exp(-\lambda\tau) \bullet \exp(-\lambda(T-\tau))]/\lambda T \bullet \exp(-\lambda T) = \tau/T$.

(e) In part (d), we found that the conditional CDF of **X** given A increases linearly from 0 at $\tau = 0$ to 1 at $\tau = T$ showing that the conditional pdf of **X** is *uniform* on (0, T] !!!

5.(a) FALSE. It is easy to verify that $f_{\mathbf{B}}(u) = (u + 1/2)$ and hence $f_{\mathbf{B},\mathbf{C}}(u,v) \ne f_{\mathbf{B}}(u)f_{\mathbf{C}}(v)$

(b) We wish to find the probability that $\mathbf{B}^2 > \mathbf{C}$. As shown in the diagram below, we have

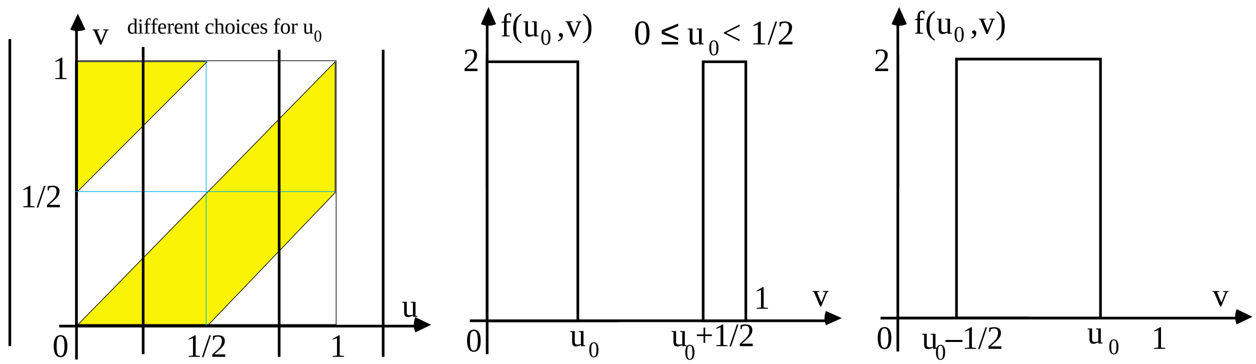
$$P\{\text{real roots}\} = \int_{u=0}^{1}\int_{v=0}^{u^2} u + v\,dv\,du = \int_{u=0}^{1} \left[uv + v^2/2\right]_{v=0}^{u^2} du = \int_{u=0}^{1} u^3 + u^4/2\,du = 0.35.$$



$$\text{Alternatively, } P\{\text{real roots}\} = \int_{v=0}^{1}\int_{u=\sqrt{v}}^{1} u + v\,du\,dv = \int_{v=0}^{1} \left[u^2/2 + uv\right]_{u=\sqrt{v}}^{1} dv = \int_{v=0}^{1} (1/2) + v/2 - v^{3/2}\,dv$$

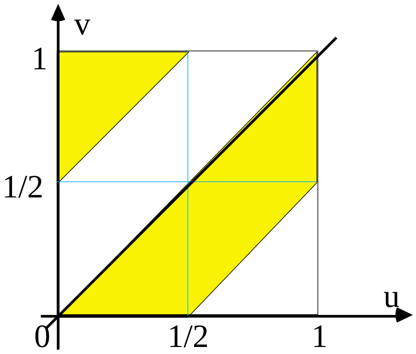
$= 0.5 + 0.25 - 0.4 = 0.35$.

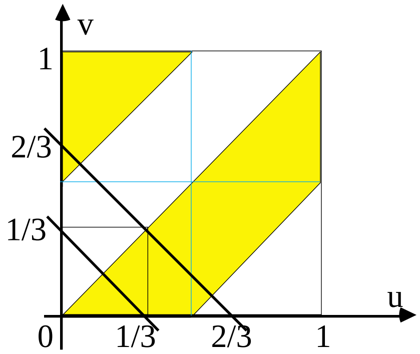
6.(a) $f_{\mathbf{X}}(u_0)$, the value of the marginal pdf at any *fixed number* u_0 is the area of the cross-section of the pdf surface at $u = u_0$, that is, $f_{\mathbf{X}}(u_0)$ equals the area under the curve $f_{\mathbf{X},\mathbf{Y}}(u_0,v)$ regarded as a function of v. As can be observed from the diagrams below, this cross-section is 0 for $u_0 < 0$ or $u_0 > 1$. On the other hand, the cross-sections look as shown below for $0 \le u_0 < 1/2$ and for $1/2 \le u_0 \le 1$.

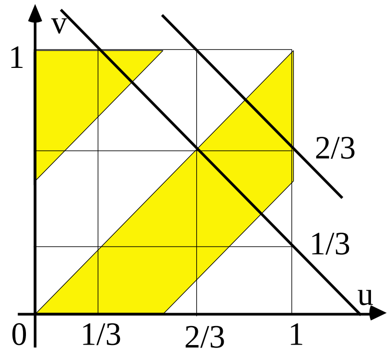


Since the cross-section areas are easily shown to be 1, it follows that $f_{\mathbf{X}}(u_0) = 1$ for all choices of u_0 in the range [0, 1] and thus $f_{\mathbf{X}}(u) = \begin{cases} 1, & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$ This is obviously a valid pdf: **X** is uniformly distributed on the interval [0,1]. A similar calculation shows that **Y** is also uniformly distributed on [0,1].

(b) $P\{\mathbf{X} > \mathbf{Y}\}$ is the probability that the random point $(\mathbf{X},\mathbf{Y})$ lies below the heavy line in the left-hand figure below. Since the joint pdf is uniform on the shaded region and three out of four dentists (oops, I mean triangles!) lie below the line, it is easily seen that $P\{\mathbf{X} > \mathbf{Y}\} = 3/4$.







(c) No, **X** and **Y** are not independent which can be proved via the "eyeball test" viz. the pdf is nonzero on a nonrectangular region, or from noting that, on the unit square, $f_{\mathbf{X}}(u)f_{\mathbf{Y}}(v) = 1 \neq f_{\mathbf{X},\mathbf{Y}}(u,v)$.

(d) $P\{\mathbf{X} + \mathbf{Y} < 1/3\}$ is the probability that the random point $(\mathbf{X},\mathbf{Y})$ lies below the heavy line (with intercepts 1/3 and 1/3 on the axes) in the middle figure below. Since the area of the shaded region is one-fourth the area of the square of side (1/3), we get that $P\{\mathbf{X} + \mathbf{Y} < 1/3\}$ = integral of joint pdf over region below line = (value of joint pdf on shaded region)•area = $2\cdot(1/4)\cdot(1/3)^2 = (1/2)\cdot(1/3)^2 = 1/18$.
Similarly, $P\{\mathbf{X} + \mathbf{Y} < 2/3\}$ is the probability that the random point $(\mathbf{X},\mathbf{Y})$ lies below the heavy line (with intercepts 2/3 and 2/3 on the axes) in the middle figure above. Since the "little extra upper piece" fits neatly into the "little empty space below", we see that
$P\{\mathbf{X} + \mathbf{Y} < 2/3\} = 4\cdot P\{\mathbf{X} + \mathbf{Y} < 1/3\} = 2/9 = (1/2)\cdot(2/3)^2$. In fact, for any real number α, $0 \le \alpha \le 1$,
$P\{\mathbf{X} + \mathbf{Y} < \alpha\} = (1/2)\cdot(\alpha)^2$. Quick check: the formula gives a probability of 0 for $\alpha = 0$ and a probability of 1/2 for $\alpha = 1$ which checks with the obvious results.
$P\{\mathbf{X} + \mathbf{Y} > 4/3\}$ is the probability that the random point $(\mathbf{X},\mathbf{Y})$ lies above the heavy line (with (unmarked) intercepts 4/3 and 4/3 on the axes) in the right-hand figure above. A little thought shows that this probability is in fact equal to $P\{\mathbf{X} + \mathbf{Y} < 2/3\} = 2/9$ and similarly
$P\{\mathbf{X} + \mathbf{Y} > 5/3\} = P\{\mathbf{X} + \mathbf{Y} < 1/3\} = 1/18$. More generally, for any real number α, $1 \le \alpha \le 2$,
$P\{\mathbf{X} + \mathbf{Y} > \alpha\} = P\{\mathbf{X} + \mathbf{Y} < 2-\alpha\} = (1/2)\cdot(2-\alpha)^2$. Quick check: the formula gives a probability of 1/2 for $\alpha = 1$ and a probability of 0 for $\alpha = 2$ which checks with the obvious results.

(e) From the results of part (d) it follows that if we set $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$, then $F_{\mathbf{Z}}(\alpha) = P\{\mathbf{Z} \le \alpha\} = P\{\mathbf{X} + \mathbf{Y} \le \alpha\}$

is given by $F_{\mathbf{Z}}(\alpha) = \begin{cases} 0, & \alpha < 0, \\ (1/2)\cdot(\alpha)^2, & 0 \le \alpha < 1, \\ 1 - (1/2)\cdot(2-\alpha)^2, & 1 \le \alpha < 2, \\ 1, & \alpha \ge 2, \end{cases}$ while $f_{\mathbf{Z}}(\alpha) = \begin{cases} \alpha, & 0 \le \alpha < 1, \\ 2-\alpha, & 1 \le \alpha < 2, \\ 0, & \text{elsewhere.} \end{cases}$

Note that for non-independent random variables, the pdf of the sum is, *in general*, ***not*** the convolution of the marginal pdfs. However, in this instance, even though **X** and **Y** are not independent, $f_{\mathbf{Z}}$ serendipitiously does happen to equal $f_{\mathbf{X}} * f_{\mathbf{Y}}$.

7.(a) $\mathbf{Z} = 5\mathbf{X} + \mathbf{Y}$ is a Gaussian random variable with mean $E[\mathbf{Z}] = E[5\mathbf{X} + \mathbf{Y}] = 5\times E[\mathbf{X}] + E[\mathbf{Y}] = 5\times 0 + 7 = 7$ and variance $\text{var}(\mathbf{Z}) = 5^2\text{var}(\mathbf{X}) + 1^2\text{var}(\mathbf{Y}) + 2\times5\times1\times\text{cov}(\mathbf{X},\mathbf{Y}) = 25\times4 + 16 + 10\rho\sqrt{\text{var}(\mathbf{X})\text{var}(\mathbf{Y})}$
$= 100 + 16 + 10\times(1/16)\times2\times4 = 121 = 11^2$. Hence, $f_{\mathbf{Z}}(w) = 1/(11\sqrt{2\pi})\cdot\exp(-(w-7)^2/242)$, $-\infty < w < \infty$.

(b) $P\{\mathbf{Y} > 3\mathbf{X}\} = P\{3\mathbf{X} - \mathbf{Y} < 0\}$. But $3\mathbf{X} - \mathbf{Y}$ is a Gaussian random variable with mean
$E[3\mathbf{X} - \mathbf{Y}] = 3\times E[\mathbf{X}] - E[\mathbf{Y}] = -7$ and variance $3^2\text{var}(\mathbf{X}) + (-1)^2\text{var}(\mathbf{Y}) + 2\times3\times(-1)\times\text{cov}(\mathbf{X},\mathbf{Y})$
$= 9\times4 + 16 - 6\rho\sqrt{\text{var}(\mathbf{X})\text{var}(\mathbf{Y})} = 36 + 16 - 6\times(1/16)\times2\times4 = 49 = 7^2$. Hence, $3\mathbf{X} - \mathbf{Y}$ is $N(-7,7^2)$ and

thus $P\{3\mathbf{X} - \mathbf{Y} < 0\} = \Phi\left(\frac{0 - (-7)}{7}\right) = \Phi(1) = 0.8413$.