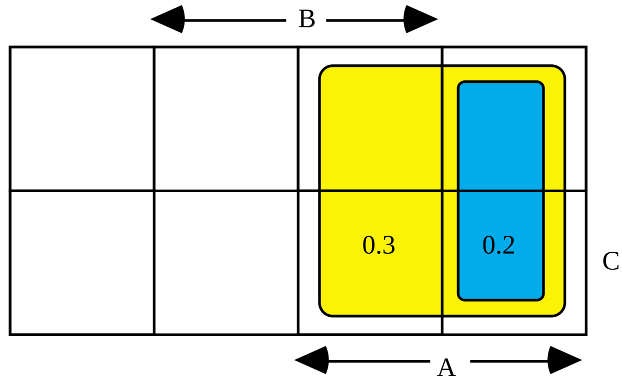
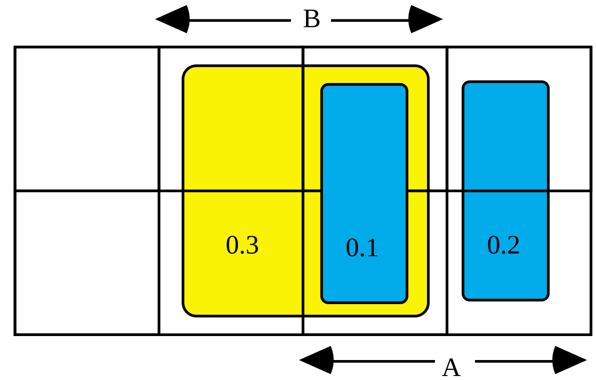
University of Illinois | **Solutions to First Hour Exam** | **ECE 313 Fall 2001**

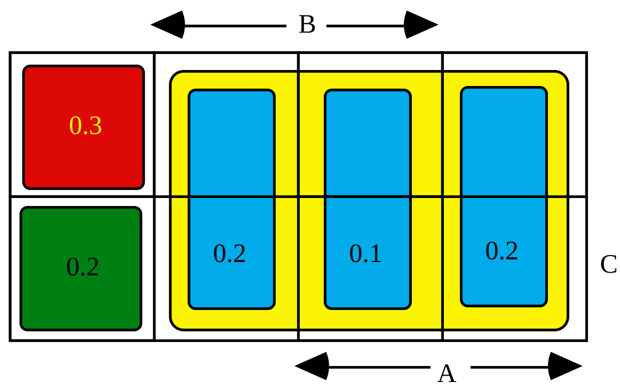
1. Given $P(A) = 0.3$, $P(B) = 0.3$, $P(C) = 0.5$, $P(A \cap B^c) = P(A^c \cap B^c \cap C) = 0.2$.

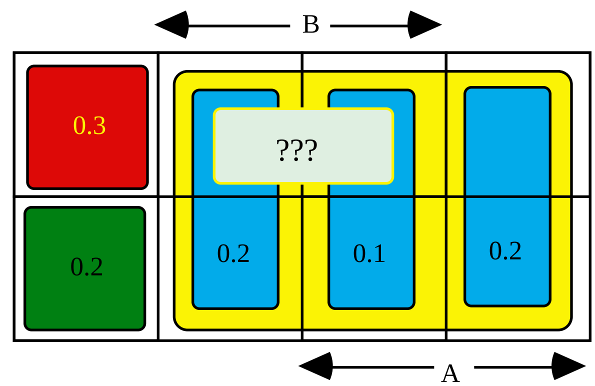
(a) A Karnaugh map/Venn diagram is very useful in solving problems such as these.





Since A is the union of the disjoint sets $A \cap B$ and $A \cap B^c$,
$P(A) = P(A \cap B) + P(A \cap B^c)$ $\quad P(A \cap B) = P(A) - P(A \cap B^c) = 0.3 - 0.2 = 0.1$.
Similarly, $P(A^c \cap B) = P(B) - P(A \cap B) = P(B) - P(A) + P(A \cap B^c) = 0.3 - 0.3 + 0.2 = 0.2$.





Next, $P(A \quad B \quad C) = P(A \quad B\ ) + P((A \quad B\ )^c \cap C) = P(A \quad B\ ) + P(A^c \cap B^c \cap C)$ (why?)
$= P(A) + P(B) - P(A \cap B) + P(A^c \cap B^c \cap C) = 0.3 + 0.3 - 0.1 + 0.2 = 0.7$,
and, of course, $P((A \quad B \quad C)^c) = 1 - P(A \quad B \quad C) = 1 - 0.7 = 0.3$. $P(B \cap C^c)$ cannot be determined.
Finally, $P(C^c \,|(A^c \cap B^c\,)) = P(A^c \cap B^c \cap C^c\,)/P(A^c \cap B^c\,)$
$= P(A^c \cap B^c \cap C^c\,)/[P(A^c \cap B^c \cap C^c\,) + P(A^c \cap B^c \cap C\,)]$
$= P((A \quad B \quad C)^c)/[P((A \quad B \quad C)^c) + P(A^c \cap B^c \cap C\,)] = 0.3/(0.3 + 0.2) = 3/5$.
Note that the value of $P(C)$ is not used anywhere except in deciding that $P(B \cap C^c)$ cannot be determined.

2.(a) $P(A \quad B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A|B)P(B)$.
Hence, $P(B) = [P(A \quad B) - P(A)]/[1 - P(A|B)] = 2/5$.

(b) $P(E|C \quad D) = \dfrac{P(E \cap (C \quad D))}{P(C \quad D)} = \dfrac{P((E \cap C) \quad (E \cap D))}{P(C \quad D)} = \dfrac{P(E \cap C) + P(E \cap D)}{P(C) + P(D)}$ since C and D are disjoint. Now, $P(E \cap C) = P(E|C)P(C)$, $P(E \cap D) = P(E|D)P(D)$, and $P(C) = 2P(D)$ giving that
$P(E|C \quad D) = \dfrac{P(E|C)P(C) + P(E|D)P(D)}{P(C) + P(D)} = \dfrac{P(E|C)2P(D) + P(E|D)P(D)}{2P(D) + P(D)} = \dfrac{2P(E|C) + P(E|D)}{3} = \dfrac{8}{21}$

3.(a) P{Fred wins first game} = P{H $\quad$ TTH $\quad$ TTTTH $\quad$ TTTTTTH $\quad$ …) $= (1/2) + (1/2)^3 + (1/2)^5 + \ldots$
$= (1/2)\bullet[1 + (1/4) + (1/4)^2 + \ldots] = (1/2)\bullet(1 - 1/4)^{-1} = 2/3$ since $(1-x)^{-1} = 1 + x + x^2 + \ldots$ for $|x| < 1$.

(b) The loser of the first game gets to go first in the next game, and thus has probability 2/3 of winning the second game. Since the tosses are independent trials, we have from the theorem of total probability that
P{Wilma wins 2nd game} = P{Wilma wins 2nd game|she loses first game}P{Wilma loses first game}
+ P{Wilma wins 2nd game|she wins first game}P{Wilma wins first game}
$= (2/3)(2/3) + (1/3)(1/3) = 5/9$.

Alternatively, consider that in a series of these games, Fred and Wilma *always* alternate in tossing the coin since the winning toss of one game is followed by the first toss of the next game (which is by the person who just lost the previous game). Hence, Wilma's wins always occur on even-numbered tosses, and the probability that she wins the second game is the probability of exactly one head in the previous n tosses, where n = 1, 3, 5, … is odd, times the probability that she tosses a head on the (n+1)th toss. Therefore,
P{Wilma wins second game} = P{head}•P{exactly one head in previous n tosses, n = 1, 3, 5, …}
= $(1/2)\cdot[1\cdot(1/2) + 3\cdot(1/2)^3 + 5\cdot(1/2)^5 + \ldots] = (1/4)\cdot[1 + 3(1/2)^2 + 5(1/2)^4 + \ldots]$
= $(1/4)\cdot[(1-1/2)^{-2} + (1+1/2)^{-2}]/2 = (1/8)\cdot[4 + 4/9] = 5/9$ where we have added together the series $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4\ldots$ and $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots$ for $|x| < 1$. It is far easier to use the theorem of total probability instead!

(c) Using Bayes' formula, P{Wilma won first game | Wilma won second game}
= P{Wilma won 2nd game|she won first game}P{Wilma won first game}/P{Wilma won second game}
= (1/9)/(5/9) = 1/5. Note that 1/9 = (1/3)(1/3) is one of the terms computed in getting to the 5/9.

(d) It is easy to see that Fred wins two games with probability (2/3)(1/3) = 2/9 while Wilma wins two games with probability (1/3)(1/3) = 1/9. They each win one game with probability (2/3)(2/3) + (1/3)(2/3) = 2/3.

Hence, $p_{\mathbf{X}}(u) = \begin{cases} 2/9 & \text{if } u = -2, \\ 2/3 & \text{if } u = 0 \\ 1/9 & \text{if } u = 2, \\ 0 & \text{for all other } u. \end{cases}$

(e) The mean value or expected value of **X** is E[**X**] = (2/9)•(–2) + (2/3)•0 + (1/9)•2 = –2/9.
We can also compute this directly as the mean winnings on first game (2/3)•(–1)+(1/3)•(+1) = –1/3 PLUS the mean winnings on the second game = (5/9)•(+1) + (4/9)•(–1) = 1/9, giving E[**X**] = –1/3 + 1/9 = –2/9.

4. **Y** = 1, 2, 3 passengers are left behind according as **X** = 6, 7, or 8. Since **X** takes on values 6, 7, 8 with with probabilities $\frac{28}{256}, \frac{8}{256}, \frac{1}{256}$ respectively, we readily find that $E[\mathbf{Y}] = \frac{1\times 28 + 2\times 8 + 3\times 1}{256} = \frac{47}{256}$.