University of Illinois at Urbana-Champaign

ECE 313: Probability with Engineering Applications

Fall 2000
Exam II

November 6, 2000

Problem 1 *(30 points)*

Using the definition of expectation:

$$\frac{2}{3} = E[X] = \int_0^1 u[a + (b-a)u]du = \frac{au}{2} + (b-a)\frac{u^3}{3}\bigg|_0^1 = \frac{a}{2} + \frac{b-a}{3} \Rightarrow a + 2b = 4$$

Since f is a density,

$$1 = \int_0^1 [a + (b-a)u]du = au + (b-a)\frac{u^2}{2}\bigg|_0^1 = a + \frac{b-a}{2} \Rightarrow a + b = 2$$

Combining these two equations we obtain

$$(2-b) + 2b = 4 \Rightarrow b = 2, \quad a = 0$$

Finally,

$$P\{X < 1/2\} = \int_0^{\frac{1}{2}} [a + (b-a)u]du = \int_0^{\frac{1}{2}} 2udu = u^2\bigg|_0^{\frac{1}{2}} = \frac{1}{4}$$

NOTE: For much of this problem, no integration is needed, since it is fairly easy to compute the corresponding areas directly using simple geometry.

Problem 2 *(30 points)*

Part (a) 5 pts. How many possible decision rules exist for this problem?

Number of decision rules = $(\text{\# of rows})^{\text{\# of cols}} = 2^3 = 8$

Part (b) 5 pts. Determine the the maximum likelihood decision rule and write it in the space below.

Sunny $\Rightarrow$ Sydney
Rain $\Rightarrow$ Atlanta
Snow $\Rightarrow$ Atlanta

Part (c) 5 pts.

Likelihood Matrix	Weather in city X		
	Sunny	Rain	Snow
Atlanta	0.24	0.52	0.24
Sydney	0.48	0.32	0.20

	Weather in city X		
	Sunny	Rain	Snow
Atlanta	0.18	0.39	0.18
Sydney	0.12	0.08	0.05

Part (d) 5 pts.

P{Sunny} = .18 + .12 = .3

Part (e) 5 pts. Assuming the same prior probabilities as in Part (c), determine the the Bayes decision rule and write it in the space below.

Sunny $\Rightarrow$ Atlanta
Rain $\Rightarrow$ Atlanta
Snow $\Rightarrow$ Atlanta

Part (f) 5 pts. Assuming the same prior probabilities as in Part (c), what is probability of error associated with the *maximum likelihood* decision rule?

P{ error } = .18 + .08 + .05 = .31

Problem 3 *(20 points)*

Part (a) 5 pts. $P\{X < 0\} = \Phi(\frac{0-60}{20}) = \Phi(-3) = 1 - \Phi(3) = 1 - 0.9987 = 0.0013$

Part (b) 5 pts. $P\{X > 100\} = 1 - \Phi(\frac{100-60}{20}) = 1 - \Phi(2) = 1 - 0.9772 = 0.0228$

Part (c) 5 pts.

$P\{50 \le X \le 70\} = \Phi(\frac{70-60}{20}) - \Phi(\frac{50-60}{20}) = \Phi(0.5) - \Phi(-0.5) = \Phi(0.5) - 1 + \Phi(0.5) = 2(0.6915) - 1 = 0.383$

Part (d) 5 pts.

$P\{50 \le X \le 60\} = \Phi(\frac{60-60}{20}) - \Phi(\frac{50-60}{20}) = \frac{1}{2} - \Phi(-0.5) = \frac{1}{2} - 1 + \Phi(0.5) = -\frac{1}{2} + 0.6915 = 0.1915$

Problem 4 *(20 points)*

Part (a) 10 pts.

The probability of success is the probability that one of the first two parallel links will function AND the middle link will function AND one of the last two parallel links will function:

$$(p + p - p^2)(p)(p + p - p^2) = \frac{3}{4}\frac{1}{2}\frac{3}{4} = \frac{9}{32}.$$

Part (b) 6 pts.

If a message can be sent, then 10 messages can be sent (any path through the network has capacity at least 10). More than 10 messages can never be sent, since the capacity of the middle link is only 10. Thus, X has two possible values, and using the results of part (a) we have

$$p_X(10) = P\{\text{success}\} = \frac{9}{32}, \qquad p_X(0) = P\{\text{failure}\} = 1 - \frac{9}{32} = \frac{23}{32}$$

Part (c) 4 pts.

Simply use the definition of expectation

$$E[X] = 0 \times p_X(0) + 10 \times p_X(10) = \frac{90}{32} = \frac{45}{16}$$