ECE 313: Probability with Engineering Applications
Fall 2000, Exam I : Solutions

Problem 1 For a geometric random variable with $n = 10$, the likelihood is given by $(1-p)^9 p$. To maximize this, set its derivative to zero:

$$\frac{d}{dp}(1-p)^9 p = (1-p)^8[1-10p] = 0 \Longrightarrow p = \frac{1}{10}.$$

The function $(1-p)^9 p$ is zero at $p = 0$ and $p = 1$, and is positive for $0 < p < 1$. Therefore, since $p = 1/10$ is the only other value for p at which the derivative is zero, this must be a maximum.

Problem 2

Part (a)

$$\frac{\binom{5}{2}}{\binom{6}{3}} = \frac{10}{20} = 1/2$$

Part (b) We will use the general version of the theorem of total probability: $P(A) = \sum_i P(A|B_i)P(B_i)$, in which the B_i partition the sample space. For this specific version of the problem, A is the event that the new monarch is a female, and the B_i are events corresponding to the possible short-lists: B_1 is the event that the short-list contains three females; B_2 is the event that the short-list contains two females and one male who is not Chuck; B_3 is the event that the short-list contains two females and Chuck; B_4 is the event that the short-list contains one female and two males who are not Chuck; B_5 is the event that the short-list contains one female and two males, one of which is Chuck. The conditional probabilities are thus: $P(A|B_1) = 1$, $P(A|B_2) = 2/3$, $P(A|B_3) = 1$, $P(A|B_4) = 1/3$, $P(A|B_5) = 1/2$. Note that there is one additional B_i, (the event that there are no females on the short-list) but since this short-list contains no females, $P(A|B_i) = 0$, and therefore it need not be included in the sum.

$$1 \times \frac{\binom{3}{3}}{\binom{6}{3}} + \frac{2}{3} \times \frac{\binom{3}{2}\binom{2}{1}}{\binom{6}{3}} + 1 \times \frac{\binom{3}{2}}{\binom{6}{3}} + \frac{1}{3} \times \frac{\binom{3}{1}\binom{2}{2}}{\binom{6}{3}} + \frac{1}{2} \times \frac{\binom{3}{1}\binom{2}{1}}{\binom{6}{3}} = 3/5$$

One could also make an argument by symmetry for this case. Chuck has been eliminated from consideration, but there is nothing in the description that singles out any of the remaining individuals for special treatment. Therefore, the remaining five must be equally likely to win. If one could show that, e.g., Beth were more likely to win than the other four, then this same argument could be applied to any of the other non-Chuck candidates, leading to a contradiction. Thus, since three of the five non-Chuck candidates are female, the probability of a female monarch is 3/5.

Part (c) Again using the theorem of total probability, there are two cases to consider: Beth and Chuck are both on the short-list, Beth is on the short-list but Chuck is not. The conditional probability of Beth being elected given that she is not on the short-list is zero, and the corresponding terms in the summation can thus be ignored.

$$\frac{1}{2} \times \frac{\binom{4}{1}}{\binom{6}{3}} + \frac{1}{3} \times \frac{\binom{4}{2}}{\binom{6}{3}} = 1/5$$

One could also use the symmetry argument from part (b) to conclude that Beth will be the new monarch with probability 1/5.

Problem 3

$$P\{X_1 \text{ and } X_2 \text{ adjacent}|X_1 < 4\} = \frac{P\{X_1 \text{ and } X_2 \text{ adjacent } \textbf{and } X_1 < 4\}}{P\{X_1 < 4\}}$$

$$\begin{aligned} P\{X_1 \text{ and } X_2 \text{ adjacent } \textbf{and } X_1 < 4\} &= P\{X_1 = 3, X_2 = 4\} + P\{X_1 = 3, X_2 = 2\} \\ &\quad + P\{X_1 = 2, X_2 = 3\} \\ &= \frac{10}{36^2} \end{aligned}$$

$$P\{X_1 < 4\} = P\{X_1 = 3\} + P\{X_1 = 2, \} = \frac{3}{36}$$

Therefore,

$$P\{X_1 \text{ and } X_2 \text{ adjacent}|X_1 < 4\} = \frac{120}{36^2} = \frac{5}{54}$$

Note: Two versions of the exam were given. On the alternate version of the exam, $X_1 > 10$, which leads to similar calculations, and the same numerical answer.

Problem 4

$$\begin{aligned} P(C|A \cup B) &= \frac{P(C \cap (A \cup B))}{P(A \cup B)} && (1) \\ &= \frac{P(CA \cup CB)}{P(A \cup B)} && (2) \\ &= \frac{P(CA) + P(CB)}{P(A) + P(B)} && (3) \\ &= \frac{P(C|A)P(A) + P(C|B)P(B)}{P(A) + P(B)} && (4) \end{aligned}$$

where (3) follows from the fact that A and B are disjoint. Therefore,

$$P(C|B) = \frac{P(C|A \cup B)[P(A) + P(B)] - P(C|A)P(A)}{P(B)} = 3/7 \text{ or } 3/14$$

depending on which version of the exam you were solving.

Problem 5

Part (a) We'll use the equation $\sigma^2 = E[X^2] - E[X]^2$. Here, X is a binomial random variable with parameters $n = 10$ and $p = 1/6$. For a binomial random variable we have

$$\sigma^2 = np(1-p) = \frac{50}{36}, \quad E[X]^2 = (np)^2 = \frac{100}{36} \Longrightarrow E[X^2] = \frac{25}{6}$$

Part (b) If you play the game N times, then

$$Z = 10X - 2(N - X) = 12X - 2N \Longrightarrow E[Z] = E[12X - 2N] = 12E[X] - 2N = 12\frac{1}{6}N - 2N = 0$$

Part (c) Here, W is a geometric random variable with parameter $p = 1/6$, therefore, $E[W] = 1/p = 6$.

Part (d) Let N_{total} be the total number of times that you play the game in order to win N times. The expected value of the winnings is given by

$$E[\text{Winnings}] = E[10N - 2(N_{total} - N)] = 10N - 2E[N_{total}] + 2N = 12N - 2\frac{N}{p} = 0$$