Bernoulli Trials / Random Variables

ECE 313
Probability with Engineering Applications
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Bernoulli Trials Example

- Consider a system with n components that requires m (≤n) or more components to function for the correct operation of the system (called m-out-of-n system). If we let m=n, then we have a series system; if we let m = 1, then we have a system with parallel redundancy.

- Assume: n components are statistically identical and function independently of each other. If we let R denote the reliability of a component (and q = 1 - R gives its unreliability), then the experiment of observing the status of n components can be thought of as a sequence of n Bernoulli trials with the probability of success equal R.

Bernoulli Trials Example (cont.)

- Now the reliability of the system is:

$$\begin{aligned} R_{m|n} &= P(\text{"}m \text{ or more components functioning properly"}) \\ &= P\Big(\bigcup_{i=m}^{n} \{\text{"exactly } i \text{ components functioning properly"}\}\Big) \\ &= \sum_{i=m}^{n} P(\text{"exactly } i \text{ components functioning properly"}) \\ &= \sum_{i=m}^{n} p(i) = \sum_{i=m}^{n} \binom{n}{i} R^i (1-R)^{n-i} \end{aligned}$$

- Remember: in terms of random variables,

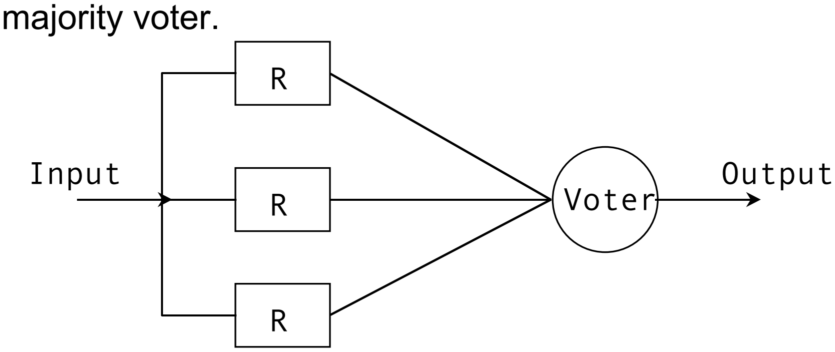
$$\binom{n}{i} R^i (1-R)^{n-i} = P(X = i)$$

where X is a r.v. representing the number of successes, and *i* is a particular value.

- It is easy to verify that: $R_{1|n} = R(parallel) = 1-(1-R)^n$ *and* $R_{n|n} = R(series) = R^n$

Bernoulli Trials
TMR System Example

- As special case of m-out-of-n system, consider a system with triple modular redundancy (TMR). In such a system there are three components, two of which are required to be in working order for the system to function properly (i.e., n = 3 and m = 2). This is achieved by feeding the outputs of the three components into a majority voter.



Bernoulli Trials
TMR System Example (cont.)

- The reliability of TMR system is given by the expression:

$$R_{TMR} = \sum_{i=2}^{3} \binom{3}{i} R^i (1-R)^{3-i} = \binom{3}{2} R^2 (1-R) + \binom{3}{3} R^3 (1-R)^0 = 3R^2(1-R) + R^3$$

- and thus

$$R_{TMR} = 3R^2 - 2R^3$$

Note that:

$$R_{TMR} \begin{cases} > R, & if\ R > \frac{1}{2} \\ = R, & if\ R = \frac{1}{2} \\ < R, & if\ R < \frac{1}{2} \end{cases}$$

- Thus TMR increases reliability over the simplex system only if the simplex reliability is greater than 0.5; otherwise decreases reliability
- Note: the voter output corresponds to a majority; it is possible for two or more malfunctioning units to agree on an erroneous vote.

Probability Mass Function

- *Probability mass function (pmf)* or the *discrete density function* of the random variable X $p_X(x)$ gives:

  the probability that the value of the random variable X obtained on a performance of the experiment is equal to x.

$$p_X(x) = P(X = x) = \sum_{X(s)=x} P(s)$$

- Properties of the pmf:
  - (p1) $0 \le p_X(x) \le 1$ for all $x \in \Re$; (since $p_X(x)$ is a probability)
  - (p2) $\sum_{x \in \Re} p_X(x) = 1$ (since the random variable assigns some value $x \in \Re$ to each sample point $s \in S$)
  - (p3) for a discrete random variable X, the set $\{x \mid p_X(x) \neq 0\}$ is a finite or countably infinite subset of real numbers. Let denote this set by $\{x_1, x_2, \ldots\}$. Then the property (p2) can be restated as: $\sum_i p_X(x_i) = 1$

Probability Distribution Function

- The probability distribution function or the cumulative distribution function (CDF) $F_X(t)$ of the random variable X is defined by:

$$F_X(t) = P(-\infty < X \le t) = P(X \le t) = \sum_{x \le t} p_X(x)$$

  where $-\infty < t < \infty$.

- It follows from this definition that:
  $P(a < X \le b) = P(X \le b) - P(X \le a) = F(b) - F(a)$
  If X is an integer-valued random variable then

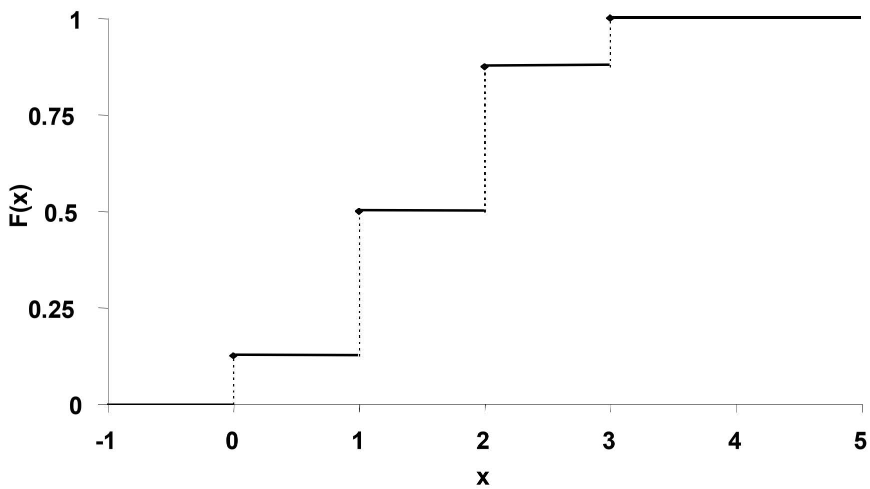
$$F(t) = \sum_{-\infty < x \le \lfloor t \rfloor} p_X(x)$$

Properties of Probability Distribution Function

- The distribution function of some discrete random variable satisfies the following properties
- **(F1)** $0 \le F(x) \le 1$ for $-\infty < x < \infty$, (F(x) is a probability)
- **(F2)** F(x) is a monotone non-decreasing function of x,i.e.,
  if $x_1 \le x_2$ then $F(x_1) \le F(x_2)$
- **(F3)** $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$
- **(F4)** F(x) has a positive jump equal to $p_X(x_i)$ at i = 1, 2, ..., and in the interval $[x_i, x_{i+1})$ F(x) has a constant value. Thus:
  $F(x) = F(x_i)$ for $x_i \le x < x_{i+1}$ and $F(x_{i+1}) = F(x_i) + p_X(x_{i+1})$

Probability Distribution Function
Example

- Consider our previous example of the sequence of three Bernoulli trials and its CDF



Discrete Distributions the Bernoulli pmf

- The Bernoulli pmf is the density function of a discrete random variable X having 0 and 1 as its only possible values and is given by:

  $p_X(0) = p_0 = P(X = 0) = q \quad p_X(1) = p_1 = P(X = 1) = p; \quad p + q = 1$

- The corresponding CDF is

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ q & \text{for } 0 \le x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$

Discrete Distributions the Binomial pmf

- The *binomial distribution* gives the probability of k “successes” in n independent trials of an experiment that has probability p of success
- Let Y_n denote the number of successes in n trials
- The value assigned to a sample point by Y_n corresponds to the number of 1’s in the n-tuple; the pmf of Y_n is

$$p_k = P(Y_n = k) = p_{Y_n}(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & \text{for } 0 \le k \le n \\ 0 & \text{otherwise} \end{cases}$$

- This equation give the probability of *k* “successes” in *n* independent trials of an experiment that has probability *p* of success on each trial.

Discrete Distributions the Binomial pmf (cont)

- Called the binomial density with parameters *n* and *p* often denoted by *b*(*k*; *n*, *p*) (e.g. *b*(*k*; 3, 0.5))
- Using the binomial theorem that:

$$\sum_{i=0}^{n} p_i = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i}$$
$$= [p+1-p]^n$$
$$= 1$$

- Typically, refer to a random variable Y_n as having binomial distribution (with parameters *n* and *p).*

Discrete Distributions the Binomial pmf (cont)

- The corresponding CDF is

$$B(t;n,p) = F_{Y_n}(t) = \sum_{i=0}^{\lfloor t \rfloor} \binom{n}{i} p^i (1-p)^{n-i}$$

Discrete Distributions the Binomial pmf (cont.)

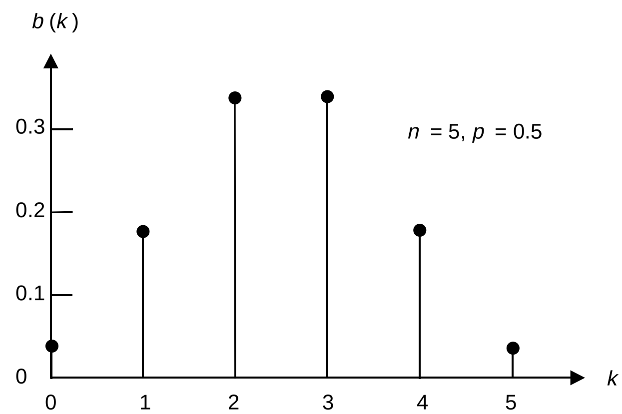
- The binomial distribution is applicable whenever a series of trials is made satisfying the following conditions:

1. Each trial has exactly two mutually exclusive outcomes *(success and failure)*

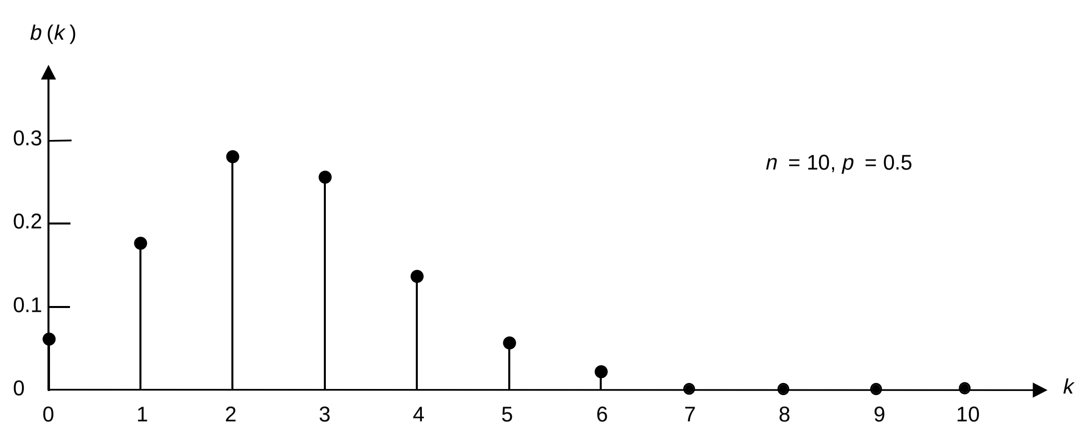
2. The probability of *success* on each trial is a constant, denoted by p. The probability of *failure* is q = 1 - p.

3. The outcomes of successive trials are mutually independent

Symmetric Binomial pmf



Positively Skewed Binomial pmf



Negatively Skewed Binomial pmf