Functions of Random Variables/Expectation and Variance

ECE 313
Probability with Engineering Applications
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Example 3

- Consider a SCSI disk drive. Assume that the number of tracks N to be traversed between two disk requests is a normally distributed random variable with mean $\mu = n/3$ and $\sigma^2 = n^2/18$ where n is the total number of tracks. The seek time T is a random variable related to the seek distance N by:

$$T = a + bN$$

- Assume that experimental data show $a = 45$, $b = 0.43$, and $n = 200$. Determine the pdf of T. Recalling that T is a nonnegative random variable whereas the normal model allows negative values, make appropriate assumptions.

Example 3 (cont.)

- We use the theorem:
- If Φ be a differentiable monotone function whose domain is I and whose range is the set of real numbers.
- Then $Y = \Phi(X)$ is a continuous random variable with the density, $f_Y(y)$ given by:

$$f_Y(y) = \begin{cases} f_X[\Phi^{-1}(y)][|(\Phi^{-1})'(y)|], & y \in \Phi(I) \\ 0, & \text{otherwise,} \end{cases}$$

Example 3 (cont)

- If $\Phi(y)$ is linear, i.e., Y = cX+d.
- Then, use of the theorem yields:

$$f_Y(y) = \begin{cases} \frac{1}{|c|} f_X\left\{\frac{y-d}{c}\right\}, & y \in cI + d, \\ 0, & \text{otherwise,} \end{cases}$$

where I is the interval over which f(x) $\neq$ 0.

Example 3 (cont.)

- Since T must be positive, we use *truncated normal density*;

$$f(x) = \begin{cases} 0 & x < 0 \\ \dfrac{1}{\alpha\sigma\sqrt{2\pi}} \exp\left[\dfrac{-(x-\mu)^2}{2\sigma^2}\right] & x \geq 0 \end{cases}$$

- Where:

$$\alpha = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(t-\mu)^2}{2\sigma^2}\right] dt$$

Example 3 (cont.)

- In our case T will be normally distributed with

$$\mu_T = a + b\frac{n}{3} = 45 + 0.43\frac{200}{3} = 73.67$$

$$\sigma_T^2 = b^2\sigma^2 = (0.43)^2\frac{n^2}{18} = 410.89$$

Example 3 (cont.)

- We show that α is close to 1:

$$\alpha = \int_0^\infty \frac{1}{\sigma_T \sqrt{2\pi}} \exp\left[\frac{-(t-\mu_T)^2}{2\sigma_T^2}\right] dt =$$

$$1 - F_Z\left(\frac{0-\mu_T}{\sigma_T}\right) = 1 - F_Z\left(\frac{-73.67}{20.27}\right) =$$

$$1 - (1 - F_Z(3.63)) = F_Z(3.63) = 0.99986$$

Example 4

- Consider a normalized floating point number in base (or radix) β so that the mantissa *X* satisfies the condition $1/\beta \leq X < 1$. Assume that the density of the continuous random variable X is $1/(x \ln \beta)$.
- (a) Show that a random deviate of *X* is given by the formula, β^{R-1} where *R* is a random number.
- (b) Determine the pdf of the normalized reciprocal $Y = 1/(\beta X)$.

Example 4 (cont.)

- It is easy to show that:

$$F_X(x) = 1 + \frac{\ln x}{\ln \beta}$$

- If R is a random number, a random deviate x of X is obtained by equating:

$$R = 1 + \frac{\ln x}{\ln \beta}$$

$$(1-R)\ln \beta = \ln \frac{1}{x}$$

$$\frac{1}{x} = e^{\ln \beta^{(1-R)}} = \beta^{1-R}$$

$$x = \beta^{R-1}$$

Example 4 (cont.)

$$F_Y(y) = P\left(\frac{1}{\beta X} \le y\right) = P\left(X \ge \frac{1}{\beta y}\right)$$

$$= 1 - F_X\left(\frac{1}{\beta y}\right) = 1 - \left(1 + \frac{\ln \frac{1}{\beta y}}{\ln \beta}\right) = \frac{\ln \beta y}{\ln \beta}$$

$$= 1 + \frac{\ln y}{\ln \beta}$$

- Hence:

$$f_Y(y) = \frac{1}{y \ln \beta}$$