Important Distributions

ECE 313 Probability with Engineering Applications Lecture 21 - November 8, 1999 Professor Ravi K. Iyer University of Illinois

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Hyperexponential Distribution

- A process with sequential phases gives rise to a hypoexponential or an Erlang distribution, depending upon whether or not the phases have identical distributions.
- If a process consists of alternate phases, i. e. during any single experiment the process experiences one and only one of the many alternate phases, **and**
- If these phases have independent exponential distributions, **then**
- The overall distribution is hyperexponential.

Hyperexporential Distripution (cont-) **Hyperexponential Distribution (cont.)**

• The density function of a *k*-phase hyperexponential random The density function of a k-phase hyperexponential random variable is:

$$
f(t) = \sum_{i=1}^{k} \alpha_i \lambda_i e^{-\lambda_i t}, \qquad t > 0, \lambda_i > 0, \alpha_i > 0, \sum_{i=1}^{k} \alpha_i = 1
$$

 \bullet • The distribution function is: The distribution function is:

$$
F(t) = \sum_i \alpha_i (1 - e^{-\lambda_i t}), \quad t \ge 0
$$

• The failure rate is: The failure rate is:

 \bullet

$$
h(t) = \frac{\sum \alpha_i \lambda_i e^{-\lambda_i t}}{\sum \alpha_i e^{-\lambda_i t}}, \qquad t > 0
$$

which is a decreasing failure rate from Σαرنج down to min {λ*1,* λ*2,…*}

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Hyperexponential Distribution (cont.) Hyperexponential Distripution (cont-)

 \bullet • The hyperexponential is a special case of mixture distributions that often arise in practice: that often arise in practice: The hyperexponential is a special case of mixture distributions

$$
F(x) = \sum_i \alpha_i F_i(x), \qquad \sum \alpha_i = 1, \alpha_i \ge 0
$$

- The hyperexponential distribution exhibits more variability than system often expresses this. the exponential, e.g. CPU service-time distribution in a computer The hyperexponential distribution exhibits more variability than system often expresses this. the exponential, e.g. CPU service-time distribution in a computer
- \bullet • If a product is manufactured in several parallel assembly lines overall product is likely to be hyperexponential. and the outputs are merged, then the failure density of the If a product is manufactured in several parallel assembly lines overall product is likely to be hyperexponential. and the outputs are merged, then the failure density of the

- α
- \bullet • The failure rate of the Weibull distribution with various values of The failure rate of the Weibull distribution warious values of
	-
	- -
	-
- **Failure Rate of the Weibull Distribution** Failure **Aate of the Weibull Distribution**
	-

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– The cumulative hazard is a power function: $H(t) =$

 \mathbf{I}

λ*t* α

– Hazard rate:

Hazard rate:

 \mathbf{I}

 $h(t) = \lambda \alpha t$

 $\lambda \alpha^{ \alpha^{-1}}$

– Distribution function:

 \mathbf{I}

 $F(t) = 1 - e^{\lambda t^{\alpha}}$

– Density:

 $f(t) = \lambda \alpha t^{\alpha-1} e$ $f(t) = \lambda \alpha t^{\alpha - 1} e^{-\lambda t}$

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Weibull Distribution

Neibril Distribution

• The Weibull distribution describes fatigue failure, vacuum-tube

The Weibull distribution describes fatigue failure, vacuum-tube

failure, and ball-bearing failure.

failure, and ball-bearing failure.

 \bullet

• It is the most widely used parametric family of failure

It is the most widely used parametric family of failure

distributions.

distributions.

• By a proper choice of its shape parameter

By a proper choice of its shape parameter α , an IFR, a DFR, or

a constant failure rate distribution can be obtained.

a constant failure rate distribution can be obtained.

• It can be used for all three phases of the mortality curve:

It can be used for all three phases of the mortality curve:

α

 $\overline{}$

, an IFR, a DFR, or

$$
\lambda = -1n(0.85)/1900 = 0.1625/1900 = 0.00008554
$$

 \bullet • Solving for Solving for λ , we get:

$$
P(X < 100 \mid X > 90) = \frac{F_X(100) - F_X(90)}{P(X > 90)}
$$
\n
$$
= \frac{F_X(100) - F_X(90)}{1 - F_X(90)}
$$
\n
$$
= \frac{e^{-\lambda(90)^2} - e^{-\lambda(100)^2}}{e^{-\lambda(90)^2}}
$$

$$
P(X < 100 \mid X > 90) = \frac{P(90 < X < 100)}{P(90 < X < 100)}
$$

Weibull Distribution Example (cont.) **Weibull Distribution Example (cont.)**

P(X < 100 | *X* $(06 < X | 001 > X)$ $=(06 < X | 001 >$ $06 < X | 001$

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ECE 313 - Fall 1999 ECE 313 - Fall 1999 **Weibull Distribution Example**

Weibull Distribution Example

• The lifetime

distribution with αdistribution with $\alpha = 2$.

The lifetime X hours of a component is modeled by a Weibull

• Starting with a large number of components, it is observed that

Starting with a large number of components, it is observed that 15 percent of the components thave lasted 90 hours fail

15 percent of the components that have lasted 90 hours fail

before 100 hours. Determine the parameter

• and

P(X

< 100|

X > 90) = 0.15

•

 $F_{\rm x}(x) = 1 - e$ $(x) = 1 - e^{\lambda x^2}$

ب

hours of a component is modeled by a Weibull

• Normal density with parameters µ =2 and

 \bullet

α
⊥

Normal or Gaussian Distribution Normal or Gaussian Distripution (cont.)

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distribution.

where

distribution.

where $-\infty < \mu < \infty$ and $\sigma > 0$ are two parameters of the

 $-\infty < \mu < \infty$ and $\sigma > 0$ are two parameters of the

 $0 < \rho$ pue ∞ $\alpha > n' > \infty$

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Normal or Ganssian Distriprinon

• Extremely important in statistical application because of the

Extremely important in statistical application because of the

– Under very general assumptions, the mean of a sample of *n*

Under very general assumptions, the mean of a sample of n

mutually independent random variables is normally distributed in

central limit theorem:

central limit theorem:

 \bullet

the limit *n*

→ ∞.

• During the wear-out phase, component lifetime follows a normal

During the wear-out phase, component lifetime follows a normal

distribution.

distribution.

• The normal density is given by:

The normal density is given by:

f x

α $f(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{2}\right)^2\right), \quad -\infty < x$

 $\frac{1}{2}$

 $\overline{}$

 $\frac{1}{\sqrt{2}}$

σ

π

 \sim \sim

 \mathcal{L}

 \sim

−∞< <∞

 $\overline{}$ \sim <u>e</u>

σ

• Errors of measurement often possess this distribution.

Errors of measurement often possess this distribution.

• The tabulations of the normal distribution are made only for *z* ≥ 0 To find *P*(*a* ≤ \overline{M} ≤ *b*, use *F(b) - F(a).*

$$
= \int_{z}^{z} \int_{z}^{z} (-t) dt
$$

= $\int_{z}^{z} f_{z}(t) dt$
= $\int_{z}^{z} f_{z}(t) dt - \int_{z}^{z} f_{z}(t) dt$
= $1 - F_{z}(z)$

- Since the standard normal density is clearly symmetric, it follows
-
- Since the standard normal density is clearly symmetric, it follows
	-
- that for *z* $\frac{v}{Q}$ *z* ∫ −∞ −

 $F_{7}(-z) = \int f_{7}(t) dt$

 $-\frac{2}{2}$

 $(-z) = \int f_z(t)$

Z f f $-$ / \sim $\sqrt{2}$

∞

-
- - **(cont.)**
-

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Normal or Gaussian Distribution

Normal or Ganssian Distriprinum

(cont.)

• The distribution function

every pair of limits *a* and

special tables.

special tables.

• These tables pertain to the

[Z~N(0,1)] --- a normal distribution with parameters

1 --- and their entries are the values of:

1 --- and their entries are the values of:

 $F_Z(z) = \frac{1}{\sqrt{2\pi}} \int e^{-iz/2} dt$

 $\frac{1}{2\pi}$ $\frac{7}{2}$

π

∫ −

 $\widetilde{(\mathcal{Z})}$ \Box

t z

 $\frac{2}{3}$

standard normal distribution

µ = 0 and

 $\frac{q}{\parallel}$

F(x) has no closed form, so between

The distribution function F(x) has no closed form, so between

b, probabilities relating to normal

distributions are usually obtained numerically and recorded in

every pair of limits a and b, probabilities relating to normal

distributions are usually obtained numerically and recorded in

Normal or Gaussian Distribution Normal or Gaussian Distripution

Normal or Ganssian Distriprition Normal or Gaussian Distribution (cont.)

 \bullet • For a particular value, corresponding value of the standardized variable *Z* is given by: *x*, of a normal random variable *X*, the

$$
z=(x-\mu)/\sigma
$$

 \bullet • The distribution function of The distribution function of X can be found by using: can be found by using:

$$
F_Z(z) = P(Z \le z)
$$

=
$$
P\left(\frac{X - \mu}{\sigma} \le z\right)
$$

=
$$
P(X \le \mu + z\sigma)
$$

=
$$
F_X(\mu + z\sigma)
$$

alternatively: alternatively:

$$
F_{\rm x}(x) = F_{\rm z}(\frac{x-\mu}{\sigma})
$$

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