Important Distributions

ECE 313
Probability with Engineering Applications
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Hyperexponential Distribution

- A process with sequential phases gives rise to a hypoexponential or an Erlang distribution, depending upon whether or not the phases have identical distributions.
- If a process consists of alternate phases, i. e. during any single experiment the process experiences one and only one of the many alternate phases, **and**
- If these phases have independent exponential distributions, **then**
- The overall distribution is hyperexponential.

Hyperexponential Distribution (cont.)

- The density function of a *k*-phase hyperexponential random variable is:

$$f(t) = \sum_{i=1}^{k} \alpha_i \lambda_i e^{-\lambda_i t}, \qquad t > 0, \lambda_i > 0, \alpha_i > 0, \sum_{i=1}^{k} \alpha_i = 1$$

- The distribution function is:

$$F(t) = \sum_i \alpha_i (1 - e^{-\lambda_i t}), \qquad \mathrm{t} \geq 0$$

- The failure rate is:

$$h(t) = \frac{\sum \alpha_i \lambda_i e^{-\lambda_i t}}{\sum \alpha_i e^{-\lambda_i t}}, \qquad \mathrm{t} > 0$$

which is a decreasing failure rate from $\Sigma\alpha_i\lambda_i$ down to min $\{\lambda_1, \lambda_2, \ldots\}$

Hyperexponential Distribution (cont.)

- The hyperexponential is a special case of mixture distributions that often arise in practice:

$$F(x) = \sum_i \alpha_i F_i(x), \qquad \sum \alpha_1 = 1, \alpha_1 \geq 0$$

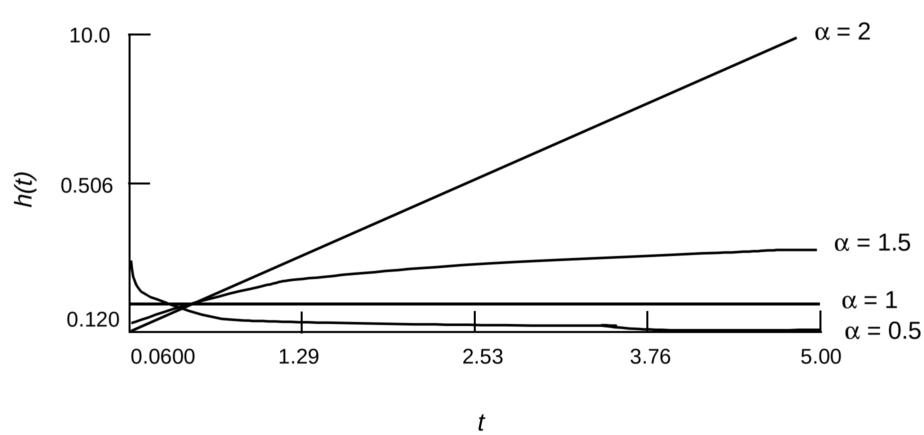
- The hyperexponential distribution exhibits more variability than the exponential, e.g. CPU service-time distribution in a computer system often expresses this.
- If a product is manufactured in several parallel assembly lines and the outputs are merged, then the failure density of the overall product is likely to be hyperexponential.

Weibull Distribution

- The Weibull distribution describes fatigue failure, vacuum-tube failure, and ball-bearing failure.
- It is the most widely used parametric family of failure distributions.
- By a proper choice of its shape parameter α, an IFR, a DFR, or a constant failure rate distribution can be obtained.
- It can be used for all three phases of the mortality curve:
  - Density: $f(t) = \lambda \alpha t^{\alpha-1} e^{-\lambda t^{\alpha}}$
  - Distribution function: $F(t) = 1 - e^{\lambda t^{\alpha}}$
  - Hazard rate: $h(t) = \lambda \alpha t^{\alpha-1}$
  - The cumulative hazard is a power function: $H(t) = \lambda t^{\alpha}$

Failure Rate of the Weibull Distribution

- The failure rate of the Weibull distribution with various values of α and $\lambda = 1$.



Weibull Distribution Example

- The lifetime X hours of a component is modeled by a Weibull distribution with $\alpha = 2$.
- Starting with a large number of components, it is observed that 15 percent of the components that have lasted 90 hours fail before 100 hours. Determine the parameter λ.
- $F_X(x) = 1 - e^{\lambda x^2}$
- and $P(X < 100 | X > 90) = 0.15$

Weibull Distribution Example (cont.)

$$P(X < 100 \mid X > 90) = \frac{P(90 < X < 100)}{P(X > 90)}$$

$$= \frac{F_X(100) - F_X(90)}{1 - F_X(90)}$$

$$= \frac{e^{-\lambda(90)^2} - e^{-\lambda(100)^2}}{e^{-\lambda(90)^2}}$$

- Solving for λ, we get:

$$\lambda = -1n(0.85)/1900 = 0.1625/1900 = 0.00008554$$

Normal or Gaussian Distribution

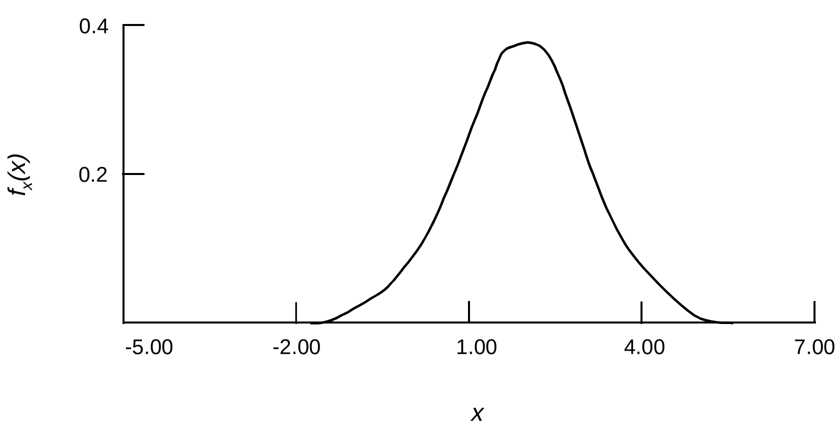
- Extremely important in statistical application because of the central limit theorem:
  - Under very general assumptions, the mean of a sample of n mutually independent random variables is normally distributed in the limit $n \to \infty$.
- Errors of measurement often possess this distribution.
- During the wear-out phase, component lifetime follows a normal distribution.
- The normal density is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right], \qquad -\infty < x < \infty$$

where $-\infty < \mu < \infty$ and $\sigma > 0$ are two parameters of the distribution.

Normal or Gaussian Distribution (cont.)

- Normal density with parameters μ =2 and σ =1



Normal or Gaussian Distribution (cont.)

- The distribution function *F(x)* has no closed form, so between every pair of limits *a* and *b*, probabilities relating to normal distributions are usually obtained numerically and recorded in special tables.
- These tables pertain to the **standard normal distribution** [Z~N(0,1)] --- a normal distribution with parameters $\mu = 0$ and $\sigma = 1$ --- and their entries are the values of:

$$F_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2} dt$$

Normal or Gaussian Distribution (cont.)

- Since the standard normal density is clearly symmetric, it follows that for *z* > 0:

$$\begin{aligned} F_Z(-z) &= \int_{-\infty}^{-z} f_Z(t)dt \\ &= \int_{z}^{\infty} f_Z(-t)dt \\ &= \int_{z}^{\infty} f_Z(t)dt \\ &= \int_{-\infty}^{\infty} f_Z(t)dt - \int_{-\infty}^{z} f_Z(t)dt \\ &= 1 - F_Z(z) \end{aligned}$$

- The tabulations of the normal distribution are made only for $z \geq 0$ To find $P(a \leq Z \leq b$, use *F(b) - F(a)*.

Normal or Gaussian Distribution (cont.)

- For a particular value, *x*, of a normal random variable *X*, the corresponding value of the standardized variable *Z* is given by:

$$z = (x - \mu) / \sigma$$

- The distribution function of *X* can be found by using:

$$\begin{aligned} F_Z(z) &= P(Z \le z) \\ &= P(\frac{X - \mu}{\sigma} \le z) \\ &= P(X \le \mu + z\sigma) \\ &= F_X(\mu + z\sigma) \end{aligned}$$

alternatively:

$$F_X(x) = F_Z(\frac{x - \mu}{\sigma})$$