Exponential Distribution and the Reliability Function

ECE 313

Probability with Engineering Applications

Lecture 17 - October 25, 1999

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The Reliability Function

- Let the random variable *X* be the lifetime or the time to failure of a component. The probability that the component survives until some time *t* is called the **reliability** R(t) of the component.
- $R(t) = P(X)t) = 1 - F(t)$
  where F is the distribution function of the component lifetime, *X*.
- The component is assumed to be working properly at time *t = 0* and no component can work forever without failure.
  i.e. $R(0) = 1 \ \ \text{limit}\ t \to \infty \ \ R(t) = 0$
- R(t) is a monotone non-increasing function of *t*.
- For *t* less than zero, reliability has no meaning, but:
  we let R(t) = 1 for *t* < 0. F(t) will often be called the **unreliability**.

The Reliability Function (cont.)

- Consider a fixed number of identical components, N_0, under test.
- After time t, $N_f(t)$ components have failed and $N_s(t)$ components have survived.
- $N_f(t) + N_s(t) = 0$.
- The estimated probability of survival:

$$\hat{P}(survival) = \frac{N_s(t)}{N_0}$$

The Reliability Function (cont.)

- In the limit as $N_0 \to \infty$, we expect $\hat{P}$ (survival) to approach $R(t)$. As the test progresses, $N_s(t)$ gets smaller and *R(t) decreases:*

$$R(t) \approx \frac{N_s(t)}{N_0}$$
$$= \frac{N_0 - N_f(t)}{N_0}$$
$$= 1 - \frac{N_f(t)}{N_0}$$

The Reliability Function (cont.)

- (N_0 is constant, while the number of failed components N_f increases with time.)
- Taking derivatives:

$$R'(t) \approx -\frac{1}{N_0} N'_f(t)$$

$N'_f(t)$ is the rate at which components fail.

- As $N_0 \to \infty$, the right hand side may be interpreted as the negative of the failure density function, $F_x(t)$:

$$R'(t) = -f_x(t)$$

- Note: $f(t)\Delta t$ is the (unconditional) probability that a component will fail in the interval (t, $t + \Delta t$].

Instantaneous Failure Rate

- If we know for certain that the component was functioning up to time t, the (conditional) probability of its failure in the interval will (in general) be different from $f(t)\Delta t$.
- This leads to the notion of "Instantaneous failure rate". Notice that the conditional probability that the component does not survive for an (additional) interval of duration x given that it has survived until time t can be written as:

$$G_t(x) = \frac{P(t < X < t + x)}{P(X > t)} = \frac{F(t + x) - F(t)}{R(t)}$$

Instantaneous Failure Rate (cont.)

- Definition: The instantaneous failure rate *h(t)* at time *t* is defined to be:

$$h(t) = \lim_{x \to 0} \frac{1}{x} \frac{F(t+x) - F(t)}{R(t)} = \lim_{x \to 0} \frac{R(t) - R(t+x)}{xR(t)}$$

so that $$h(t) = \frac{f(t)}{R(t)}$$

- *h(t)Δt* represents the conditional probability that a component surviving to age *t* will fail in the interval (*t, t* + *Δt*].
- The exponential distribution is characterized by a constant instantaneous failure rate:

$$h(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda$$

Instantaneous Failure Rate (cont.)

- Integrating both sides of the equation:

$$\int_0^t h(x)dx = \int_0^t \frac{f(x)}{R(x)}dx$$

$$= \int_0^t -\frac{R'(x)}{R(x)}dx$$

$$= -\int_{R(0)}^{R(t)} \frac{dR}{R}$$

or $$-1nR(t) = \int_0^t h(x)dx$$

(Using the boundary condition, R(0) = 1) Hence:

$$R(t) = \exp\left[-\int_0^t h(x)dx\right]$$

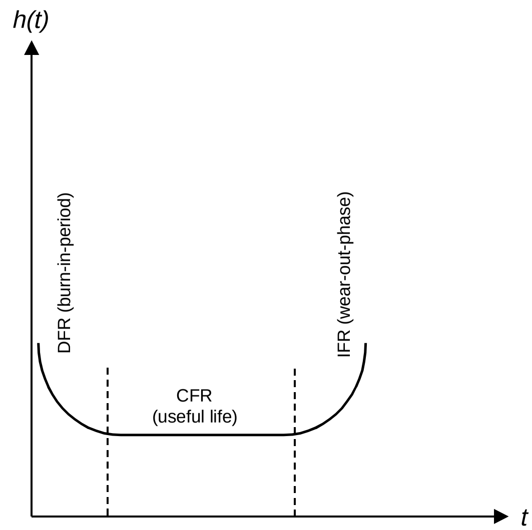
Cumulative Hazard

- The cumulative failure rate, $H(t) = \int_0^t h(x)dx,$ is referred to as the **cumulative hazard**.
- $R(t) = \exp\left[-\int_0^t h(x)dx\right]$ gives a useful theoretical representation of reliability as a function of the failure rate.
- An alternate representation gives the reliability in terms of cumulative hazard: $R(t) = e^{-H(t)}$.
- If the lifetime is exponentially distributed, then $H(t) = \lambda t$ and we obtain the exponential reliability function.

f(t) and h(t)

- $f(t)\Delta t$ is the unconditional probability that the component will fail in the interval $(t, t + \Delta t]$.
- $h(t)\Delta t$ is the conditional probability that the component will fail in the same time interval, given that it has survived until time t.
- $h(t)$ is always greater than or equal to $f(t)$, because $R(t) \leq 1$.
- $f(t)$ is a probability density. $h(t)$ is not.
- [*h(t)*] is the failure rate.
- [*f(t)*] is the failure density.
- To further see the difference, we need the notion of conditional probability density.

Failure Rate as a Function of Time



Treatment of Failure Data

- Part failure data generally obtained from two sources: the failure times of various items in a population placed on a life test, or repair reports listing operating hours of replaced parts in equipment already in field use.
- Compute and plot either the failure density function or the instantaneous failure rate as a function of time.
- The data: a sequence of times to failure, but the failure density function and the hazard introduced as continuous variables.
- Compute a piecewise-continuous failure density function and hazard rate from the data.
- This is, a specific approach to the very general engineering problem of how to model a problem from certain qualitative knowledge about the system supported by quantative data.

Treatment of Failure Data (cont.)

- Define piecewise-continuous failure density and hazard-rate in terms of the data.
- Assume that our data describe a set of N items placed in operation at time t=0. As time progresses, items fail, and at any time t the number of survivors is n(t).
- The empirical probability density function defined of over the time interval $t_i < t \le t_i + \Delta t_i$, is given by the ratio of the number of failures occurring in the interval to the *size of the original population*, divided by the length of the time interval:

$$f_d(t) = \frac{[n(t_i) - n(t_i + \Delta t_i)] / N}{\Delta t_i} \quad \text{for } t_i < t \le t_i + \Delta t_i$$

Treatment of Failure Data (cont.)

- The data hazard (inst.failure rate) over the interval $t_i < t \le t_i + \Delta t_i$ is the ratio of the number of failures occurring in the time interval to the *number of survivors at the beginning of the time interval*, divided by the length of the time interval:

$$z_d(t) = \frac{[n(t_i) - n(t_i + \Delta t_i)] / n(t_i)}{\Delta t_i} \quad \text{for } t_i < t \le t_i + \Delta t_i$$

- Observation: The failure density function $f_d(t)$ is a measure of the *overall speed* at which failures are occurring, whereas the hazard rate $z_d(t)$ is a measure of the *instantaneous speed* of failure.
- Note: Both $f_d(t)$ and $z_d(t)$ have the dimensions of inverse time (generally the time unit is hours).
- The choice of t_i and Δt_i in the above equations is unspecified and is best discussed in terms of the examples that follow.

Properties of Density and Distribution Functions

No.	Distribution Function	Density Function
1.	F(x) for $x_1 < x \le x_2$ Distribution function defined over range $x_1 < x \le x_2$	f(x) for $x_1 < x \le x_2$ Distribution function defined over range $x_1 < x \le x_2$
2.	$P(a < x \le b) = F(b) - F(a)$ Probability that x lies between a and b	$P(a < x \le b) = \int_a^b f(x)dx$ Probability that x lies between a and b
3.	F(x) cannot decrease as x increases	$f(x) \ge 0$ f(x) is never negative
4.	$F(x_1) = 0$ and $F(x_2) = 1$ Probability ranges from 0 to 1	$\int_{x_1}^{x_2} f(x(dx) = 1$ Probability of the sample space is unity

Constraints on f(t) and z(t)

No.	Density Function	Hazard Rate
1.	f(t) for $0 < t \le \infty$ Density function is defined for all positive time	z(x) for $0 < t \le \infty$ Hazard rate is defined for all positive time
2.	$f(t) \ge 0$ f(t) is never negative	$z(t) \ge 0$ z(t) is never negative
3.	$\int_0^\infty f(t)dt = 1$ Probability of sample space is unity	$\int_0^\infty z(t)dt = \infty$ Equivalent to condition on f(t)